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Random Walk Intersections

Large Deviations and Related Topics

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RANDOM WALK INTERSECTIONS: LARGE DEVIATIONS AND SOME RELATED TOPICS

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Preface

This book aims to provide a systematic account for some recent progress on the large deviations arising from the area of sample path intersections, including the calculation of the tail probabilities of intersection local times, ranges of Brownian motions and random walks.

Quantities measuring the amount of self-intersection of a random walk, or of mutual intersection of several independent random walks have been studied intensively for more than twenty years, see *e.g.* [54], [56], [117], [118], [109], [20], [124], [82], [127][128], [15], [86], [10], [9], [107]. This research is often motivated by the role these quantities play in renormalization group methods for quantum field theory, see *e.g.* [74], [48], [49], [62]; in our understanding of polymer models, see *e.g.* [126], [17],[91], [155], [158], [159],[160],[61], [99], [19], [89], [88]; or in the analysis of stochastic processes in random environments, see *e.g.* [100], [104],[39], [40] [78], [90], [1], [38] [75], [79].

Sample path intersection is also an important subject within the probability comuunity. It has been known ([45], [130], [46]) that sample path intersections has a deep link to the problems of cover times and thick points through tree-encoding techiniques. Finally, it is impossible to write a book on sample path intersection without mentioning the influential work led by Lawler, Scheramm and Werner ([111], [112], [113], [110]) on the famous intersection exponent problem and on other Brownian sample path properties in connection to the Stochastic Loewner Evolution, which counts as one of the most exciting developments made in the fields of probability in the recent years.

In contrast to the behavior patterns investigated by Lawler, Scheramm and Werner, where the sample paths avoid each other and are loop-free, most part of this book concerns about the probabilities that the random walks and Brownian motions intersect each other or themselves with extreme intensities. When these probabilities decay with exponential rates, the problem falls into the category of large deviations. In recent years, there has been some substantial input of the new tools and new ideas to this subject. The list includes the method of high moment asymptotics, sub-additivity created by moment inequality, and the probability in Banach space combined with Feynman-Kac formula. Correspondent to the progress in methodology, the established theorems have been accumulated into a rather complete picture of the area. These developments make it desirable to write a monograph on this subject which has not been adequately exposed in a systematic way.

This book is transformed form the lecture notes for a year-long graduate course at University of Tennessee in the school year 2006-2007. Making it accessible to the non-experts with the basic knowledge of stochastic processes and functional analysis has been one of my guidelines in writing it. To make it reasonably self-contained, I add Chapter 1 for the general theory of the large deviations. Most of theorems listed in this chapter are not always easy to tracked down in literature. In addition, a small amount of Exercises are included in the "Notes and comments" section in each chapter, an effort to promote active reading. Some of them were adapted from literature and others were created as the "second option" or "formal first option" to the problems in the course of the research activity. There is no logic connection to the later development. Consequently, skipping any exercise has no direct impact on reading the book.

The topics and results included in the book do reflect my taste and my involvement on the subject. The "Notes and comments" section in the end of each chapter is part of the effort to counter-balance the resulted partiality. Some relevant works not included in the other sections may appear here. In spite that, I would like to apologize in advance for possible inaccuracy in historic perspective appearing in the book.

In the process of investigating the subject and writing the book, I benefited from the help of several people. It is my great pleasure to acknowledge the contributions, which appear throughout the whole book, made by my collaborators R. Bass, W. Li, P. Mörters and J. Rosen, in the course of several year's collaboration. I would like to express my special thanks to D. Khoshnevisan, from whom I learned for the first time the story about intersection local times. I thank A. Dembo, J. Denzler, A. Dorogovtsev, B. Duplantier, X. Feng, S, Kwapien, J. Rosinski, A. Freire, J-F. Le Gall, D. Wu, M. Yor for discussion, information and encouragement.

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Contents

Chapter 1

Basics on large deviations

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In this chapter we introduce some general theorems on large deviations which will be frequently used in this book. In most of the cases, the state space we deal with is the real line. Indeed, a substantial portion of the discussion is limited to the random variables taking non-negative values. Sometimes, the underline stochastic processes are sub-additive (see Section 1.3). Unlike most textbooks on this subject, we put more attention on the tail probability

$$\mathbb{P}\{Y_n \geq \lambda\}$$

than the probability of the form

$$\mathbb{P}\{Y_n \in A\}.$$

The unique structure of the models we deal with in this book requires some non-conventional treatments. The topics we choice in this chapter reflect this demand. As a consequence, most theorems introduced in Section 1.2 and in Section 1.3 are non-standard and are not usually seen in the textbooks of large deviations.

1.1 Gärtner-Ellis' theorem

In the field of large deviations, people concern about asymptotic computation of small probabilities on an exponential scale. The general form of large deviation can be roughly described as

$$\mathbb{P}\{Y_n \in A\} \approx \exp\{-b_n I(A)\} \qquad (n \to \infty)$$

for a random sequence $\{Y_n\}$, a positive sequence $\{b_n\}$ with $b_n \to \infty$, and a coefficient $I(A) \geq 0$. In the application, we often concern about the probability that the random variable(s) takes large values. Since the remarkable works by Donsker-Varadhan (and others) in seventies and eighties, the field has been developed into a relatively complete system. There have been several standard approaches in dealing with large deviation problems. Perhaps the most useful tool is Gätner-Ellis Theorem.

We have no intention to state the large deviation theory in its full generality. Let $\{Y_n\}$ be a sequence of real random variables and let $\{b_n\}$ be a positive sequence such that $b_n \longrightarrow \infty$.

Assumption 1.1.1 *For each $\theta \in \mathbb{R}$, the logarithmic moment generating function $\Lambda(\theta)$, defined as the limit*

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta b_n Y_n\Big\} \qquad \theta \in \mathbb{R} \tag{1.1.1}$$

exists as an extended real number. Further, the origin belongs to the interior $\mathcal{D}_\Lambda^o$ of the domain $\mathcal{D}_\Lambda = \{\theta \in \mathbb{R};\ \ \Lambda(\theta) < \infty\}$ of the function $\Lambda(\theta)$.

By Hölder inequality, $\Lambda(\theta)$ is a convex function. Consequently, $\mathcal{D}_\Lambda^o = (a, b)$ for some $-\infty \leq a < 0 < b \leq \infty$ and $\Lambda(\theta)$ is continuous in its domain $\mathcal{D}_\Lambda$. Define the *Fenchel-Legendre transform* $\Lambda^*(\lambda)$ of $\Lambda(\theta)$ as

$$\Lambda^*(\lambda) = \sup_{\theta>0} \Big\{\theta\lambda - \Lambda(\theta)\Big\} \qquad \lambda \in \mathbb{R}. \tag{1.1.2}$$

To discuss the role played by the function $\Lambda^*(\cdot)$, we introduce the following definition

Definition 1.1.2 . *A function I: $\mathbb{R} \longrightarrow [0, \infty]$ is called a rate function , if it is lower semi-continuous: For any $l > 0$, the level set*

$$I_l = \{\lambda \in \mathbb{R};\ \ I(\lambda) \leq l\}.$$

is a close set. Further, a rate function is said to be good, if every level set is compact in $\mathbb{R}$.

We point out an equivalent statement of lower semi-continuity: For any $\lambda_n, \lambda \in \mathbb{R}$ with $\lambda_n \to \lambda$,

$$\liminf_{n\to\infty} I(\lambda_n) \geq I(\lambda). \tag{1.1.3}$$

We now claim that under Assumption 1.1.1, the function $\Lambda^*(\cdot)$ is a good rate function . Indeed, by definition

$$\Lambda^*(\lambda_n) \geq \theta\lambda_n - \Lambda(\theta) \qquad \theta \in \mathbb{R} \quad n = 1, 2, \cdots$$

which leads to

$$\liminf_{n\to\infty} \Lambda^*(\lambda_n) \geq \theta\lambda - \Lambda(\theta) \qquad \theta \in \mathbb{R}.$$

Taking the supremum over $\theta \in \mathbb{R}$ on the right hand side proves (1.1.3).

Let $l > 0$ be fixed. By Assumption 1.1.1 and by the continuity of $\Lambda(\theta)$ in its domain there is $\delta > 0$ such that

$$c \equiv \sup_{|\theta|\leq\delta} \Lambda(\theta) < \infty.$$

Consequently, for any λ with $\Lambda^*(\lambda) \leq l$,

$$l \geq \sup_{|\theta|\leq\delta} \Big\{\theta\lambda - \Lambda(\theta)\Big\} \geq \delta|\lambda| - c.$$

Therefore, the level set

$$\Lambda^*_a = \{\lambda \in \mathbb{R}, \quad \Lambda^*(\lambda) \leq l\}$$

is compact.

In addition, the fact that $\Lambda^*(\lambda)$ is the conjugate of the convex function $\Lambda(\theta)$ makes $\Lambda^*(\lambda)$ a convex function.

Definition 1.1.3 *A convex function Λ: $\mathbb{R} \longrightarrow (-\infty.\infty]$ is essentially smooth if:*

(1) $\mathcal{D}^o_\Lambda = (a, b)$ is non-empty.

(2) $\Lambda(\theta)$ is differentiable in $\mathcal{D}^o_\Lambda$.

(3) $\Lambda(\cdot)$ is steep:

$$\lim_{\theta\to a^+} \Lambda'(\theta) = \lim_{\theta\to b^-} \Lambda'(\theta) = \infty$$

We introduce our first main theorem without proof, for which the reader is referred to, for example, Theorem 2.3.6, [44].

Theorem 1.1.4 *Let Assumption 1.1.1 hold. For any close set $F \in \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \in F\} \leq -\sup_{\lambda\in F} \Lambda^*(\lambda). \tag{1.1.4}$$

If we further assume that the logarithmic moment function $\Lambda(\theta)$ is essentially differentiable then for any open set $G \in \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \in G\} \geq - \sup_{\lambda\in G} \Lambda^*(\lambda). \tag{1.1.5}$$

Theorem 1.1.4 is known as *Gärtner-Ellis's theorem* on large deviations. The limit form described in (1.1.4) and (1.1.5) is called *large deviation principle* (LDP) in literature. In application, our attention mainly focus on the event that the random variables deviate away from its equilibrium state, or take large values. In the following we consider a historically important example. Let $\{X_k\}_{k\geq 1}$ be a real independent and identically distributed (i.i.d.) sequence such that there is a $c > 0$ such that

$$\mathbb{E}\exp\big\{c|X_1|\big\} < \infty. \tag{1.1.6}$$

If we take Y_n to be the sample average

$$\overline{X}_n = \frac{1}{n}\big\{X_1 + \cdots + X_n\big\} \qquad n = 1, 2\cdots. \tag{1.1.7}$$

$b_n = n$ then all assumptions given in Theorem 1.1.4 are satisfied with

$$\Lambda(\theta) = \log \mathbb{E}\exp\big\{\theta X_1\big\}.$$

Consequently, (1.1.4) and (1.1.5) holds. Observe that by Jensen inequality

$$\log \mathbb{E}\exp\big\{\theta X_1\big\} \geq \mathbb{E}X_1.$$

Hence, $\Lambda^*(\mathbb{E}X_1) = 0$. On the other hand, assume that $\lambda \in \mathbb{R}$ satisfies $\Lambda^*(\lambda) = 0$. By definition

$$\lambda\theta \leq \log \mathbb{E}\exp\big\{\theta X_1\big\} \qquad \theta \in \mathbb{R}.$$

By the fact that

$$\lim_{\theta\to 0} \theta^{-1} \log \mathbb{E}\exp\big\{\theta X_1\big\} = \mathbb{E}X_1,$$

letting $\theta \to 0^+$ and letting $\theta \to 0^-$ give, respectively, $\lambda \leq \mathbb{E}X_1$ and $\lambda \geq \mathbb{E}X_1$.

Summarizing our argument, $\Lambda^*(\lambda) = 0$ if and only if $\lambda = \mathbb{E}X_1$. By the goodness of the rate function $\Lambda^*(\cdot)$, therefore, for given $\epsilon > 0$,

$$\inf_{|\lambda-\mathbb{E}X_1|\geq\epsilon} \Lambda^*(\lambda) > 0.$$

This observation shows that the probability that the sample average deviates away from the sample mean $\mathbb{E}X_1$ has a genuine exponential decay. We summarize our observation into the following theorem known as *Cramér's large deviation principle.*

Theorem 1.1.5 *Under the assumptions (1.1.6),*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\{\overline{X}_n\in F\}\le -\inf_{x\in F}\Lambda^*(x)$$

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\{\overline{X}_n\in G\}\ge -\inf_{x\in G}\Lambda^*(x)$$

for any close set $F\subset\mathbb{R}$ and any open set $G\subset\mathbb{R}$.

In particular, for any $\epsilon>0$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{|\overline{X}_n-\mathbb{E}X_1|\ge\epsilon\Big\}<0.$$

For the inverse of Gärtner-Ellis theorem, we state the following *Varadhan's integral lemma.* For the proof, the reader is referred to Theorem 4.3.1, [44].

Theorem 1.1.6 *Let $\Lambda^*(\cdot)$ be a good rate function and let $Q(\lambda)$ be a continuous function on $\mathbb{R}$*

(1). Assume that (1.1.5) hold for every open set G. Then

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\big\{b_nQ(Y_n)\big\}\ge\sup_{\lambda\in\mathbb{R}}\Big\{Q(\lambda)-\Lambda^*(\lambda)\Big\}.$$

(2). Assume that (1.1.4) hold for every close set F and that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\big\{(1+\epsilon)b_nQ(Y_n)\big\}<\infty$$

for some $\epsilon>0$. Then for every $\theta\in\mathbb{R}$,

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\big\{\theta b_nQ(Y_n)\big\}\le\sup_{\lambda\in\mathbb{R}}\Big\{Q(\lambda)-\Lambda^*(\lambda)\Big\}.$$

The general theory of large deviations has been extended to the random variables taking values in abstract topological spaces. If Y_n is a sequence of random variables taking values in a separable Banach space B, for example, to extend Gärtner-Ellis theorem we assume the existence of the limit

$$\Lambda(f)\equiv\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{b_nf(Y_n)\Big\}\qquad f\in B^*$$

instead of (1.1.1), where B^* is the topological dual of B. In addition to some smoothness assumptions on $\Lambda(f)$, it is required that $\{Y_n\}$ be *exponentially tight* : For any $l>0$ there is a compact set $K\subset B$ such that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\{Y_n\notin K\}\le -l.$$

In dealing with the large deviations in infinite dimensional spaces, the main challenge often lies in the issue of exponential tightness. In the rest of the section we introduce a result of de Acosta [41].

Recall that a set $K \subset B$ is said to be *positively balanced*, if $\lambda x \in K$ whenever $x \in K$ and $0 \le \lambda \le 1$. *The Minkowski functional* of a convex and positively balanced set K is defined by

$$q_K(x) = \inf\{\lambda > 0; \quad x \in \lambda K\}$$

with customary convention that $\inf \varphi = \infty$. The Minkowski functional $q_K(\cdot)$ is sub-additive and positively homogeneous:

$$q_K(x+y) \le q_K(x) + q_K(y) \quad \text{and} \quad q_K(\lambda x) = \lambda q_K(x) \tag{1.1.8}$$

where $x, y \in B$ and $\lambda \ge 0$.

A family $\{\mu_\alpha; \quad \alpha \in \Theta\}$ of probability measures on B is said to be *uniformly tight*, if for any $\epsilon > 0$, there is a compact set $K \subset B$ such that

$$\mu_\alpha(K) \ge 1 - \epsilon \qquad \alpha \in \Theta.$$

The following result is given in Theorem 3.1, [41]. We state it without proof.

Theorem 1.1.7 *Let $\{\mu_\alpha, \quad \alpha \in \Theta\}$ be a family of probability measures on the separable Banach space B and assume that $\{\mu_\alpha; \quad \alpha \in \Theta\}$ is uniformly tight and that*

$$\sup_{\alpha\in\Theta} \int_B \exp\big\{\lambda ||x||\big\} \mu_\alpha(dx) < \infty \qquad \forall \lambda > 0.$$

There is a convex, positively balanced and compact set $K \subset B$ such that

$$\sup_{\alpha\in\Theta} \int_B \exp\big\{q_K(x)\big\} \mu_\alpha(dx) < \infty.$$

1.2 LDP for non-negative random variables

In this section we assume that $\{Y_n\}$ take non-negative values. Recall that the full large deviation principle is stated as: For every close set $F \subset \mathbb{R}^+$,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{Y_n \in F\big\} \le -\inf_{\lambda\in F} I(\lambda) \tag{1.2.1}$$

and, for every open set $G \subset \mathbb{R}^+$

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{Y_n \in G\big\} \ge -\inf_{\lambda\in G} I(\lambda). \tag{1.2.2}$$

In application, (1.2.1) and (1.2.2) are often replaced by our concern of the tail probability of the form

$$\mathbb{P}\{Y_n \ge \lambda\} \qquad \lambda > 0.$$

Under some mild conditions on the rate function $I(\lambda)$, the following theorem shows that large deviation principle is determined by the asymptotic behavior of tail probabilities.

Theorem 1.2.1 *Assume that the rate function $I(\lambda)$ is restrictively increasing and continuous in the interior $\mathcal{D}_I^o$ of its domain $\mathcal{D}_I = \{\lambda \in \mathbb{R}^+;\ \ I(\lambda) < \infty\}$. The following two statements are equivalent:*

(1). The large deviation principle stated by (1.2.1) and (1.2.2) holds for, respectively, every close set $F \subset \mathbb{R}^+$ and every open set $G \subset \mathbb{R}^+$.

(2). For every $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{Y_n \ge \lambda\big\} = -I(\lambda). \tag{1.2.3}$$

Proof. Clearly, (1) implies (2) under the monotonicity and continuity of the function $I(\cdot)$.

Assume that (2) holds. For a close set $F \subset \mathbb{R}^+$, let $\lambda_0 = \inf F$. We have that

$$\mathbb{P}\{Y_n \in F\} \le \mathbb{P}\{Y_n \ge \lambda_0\}.$$

By (1.2.3) we have

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{Y_n \in F\big\} \le -I(\lambda_0) = -\inf_{\lambda\in F} I(\lambda)$$

where the last step follows from monotonicity of $I(\cdot)$.

To establish (1.2.2), let $\lambda_0 \in G$. There is a $\delta > 0$ such that $[\lambda_0, \lambda_0 + \delta) \subset G$. By the fact that

$$\mathbb{P}\{Y_n \ge \lambda_0\} \le \mathbb{P}\{Y_n \ge \lambda_0 + \delta\} + \mathbb{P}\{Y_n \in [\lambda_0, \lambda_0 + \delta)\}$$

and (1) we have

$$-I(\lambda_0) \le \max\bigg\{ -I(\lambda_0 + \delta), \quad \liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \in [\lambda_0, \lambda_0 + \delta)\}\bigg\}.$$

By assumption we have that $I(\lambda_0+\delta) > I(\lambda_0)$. Consequently,

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\big\{Y_n\in G\big\}\ge -I(\lambda_0).$$

Taking supremum over $\lambda_0\in G$ on the right hand side gives (1.2.2). $\square$

The following theorem shows that under certain condition, the tail probability of the fixed sum of independent non-negative random variables is dominated by the tail probabilities of individual terms.

Theorem 1.2.2 *Let $Z_1(n),\cdots,Z_l(n)$ be independent non-negative random variables with $l\ge 2$ being fixed.*

(a). If there are constant $C_1>0$ and $0<a\le 1$ such that

$$\limsup_{n\to\infty}\frac{1}{b_n}\mathbb{P}\big\{Z_j(n)\ge\lambda\big\}\le -C_1\lambda^a\qquad\forall\lambda>0$$

for $j=1,\cdots,l$, then

$$\limsup_{n\to\infty}\frac{1}{b_n}\mathbb{P}\big\{Z_1(n)+\cdots+Z_l(n)\ge\lambda\big\}\le -C_1\lambda^a\qquad\forall\lambda>0.$$

(b) If there are constant $C_2>0$ and $b\ge 2$ such that

$$\limsup_{n\to\infty}\frac{1}{b_n}\mathbb{E}\exp\Big\{\theta b_n Z_j(n)\Big\}\le C_2\theta^b\qquad\forall\theta>0$$

for $j=1,\cdots,l$, then

$$\limsup_{n\to\infty}\frac{1}{b_n}\mathbb{E}\exp\Big\{\theta b_n\sqrt{Z_1^2(n)+\cdots+Z_l^2(n)}\Big\}\le C_2\theta^b\qquad\forall\theta>0.$$

Proof. Clearly, part (a) needs only to be proved in the case $l=2$. Given $0<\delta<\lambda$, let $0=x_0<x_1<\cdots<x_N=\lambda$ be a partition of $[0,\lambda]$ such that $x_k-x_{k-1}<\delta$. Then

$$\mathbb{P}\big\{Z_1(n)+Z_2(n)\ge\lambda\big\}\le\sum_{k=1}^N\mathbb{P}\{Z_1(n)\ge x_{k-1}\}\mathbb{P}\{Z_2(n)\ge\lambda-x_k\}.$$

Consequently,

$$\limsup_{n\to\infty}\frac{1}{b_n}\mathbb{P}\big\{Z_1(n)+Z_2(n)\ge\lambda\big\}\le -C_1\min_{1\le k\le N}\Big\{x_{k-1}^a+(\lambda-x_k)^a\Big\}.$$

By the fact $0 < a \le 1$, $x_{k-1}^a + (\lambda - x_k)^a \ge (\lambda - x_k + x_{k-1})^a \ge (\lambda - \delta)^a$. So we have

$$\limsup_{n\to\infty} \frac{1}{b_n} \mathbb{P}\big\{Z_1(n) + Z_2(n) \ge \lambda\big\} \le -C_1(\lambda - \delta)^a.$$

Letting $\delta \to 0^+$ on the right hand side proves the part (a).

We now comes to part (b). For any $\epsilon > 0$, there are finite many $\mathbf{t}_j = (t_{j,1}, \cdots, t_{j,l}) \in (\mathbb{R}^+)^l$ $j = 1, \cdots, N$ such that

$$\sqrt{t_{j,1}^2 + \cdots + t_{j,l}^2} = 1 \qquad j = 1, \cdots, N,$$

$$\sqrt{x_1^2 + \cdots + x_l^2} \le (1+\epsilon) \max_{1\le j\le N} \sum_{k=1}^{l} t_{j,k} x_k \qquad \forall x_1, \cdots, x_l \in (\mathbb{R}^+)^l.$$

Hence,

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\theta b_n \sqrt{Z_1^2(n) + \cdots + Z_l^2(n)}\Big\} \\
&\le \sum_{j=1}^{N} \mathbb{E}\exp\Big\{(1+\epsilon)\theta b_n \sum_{k=1}^{l} t_{j,k} Z_k(n)\Big\} \\
&= \sum_{j=1}^{N} \prod_{k=1}^{N} \mathbb{E}\exp\Big\{(1+\epsilon)\theta b_n t_{j,k} Z_k(n)\Big\}.
\end{aligned}$$

Hence, by the fact that $b \ge 2$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{b_n} \mathbb{E}\exp\Big\{\theta b_n \sqrt{Z_1^2(n) + \cdots + Z_l^2(n)}\Big\} \\
&\le C_2(1+\epsilon)^b \theta^b \max_{1\le j\le N} \sum_{k=1}^{N} t_{j,k}^b \le C_2(1+\epsilon)^b\theta^b.
\end{aligned}$$

Letting $\epsilon \to 0^+$ on the right hand side proves the part (b). □

The following theorem appears as a version of Gr̈tner-Ellis large deviation.

Theorem 1.2.3 *Assume that for all $\theta \ge 0$, the limit*

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\Big\{\theta b_n Y_n\Big\} \tag{1.2.4}$$

exists as an extended real number, and that the function $\Lambda(|\theta|)$ is essentially smooth.

(1). The function

$$I(\lambda) = \sup_{\theta>0} \Big\{\theta\lambda - \Lambda(\theta)\Big\} \qquad \lambda \ge 0 \tag{1.2.5}$$

is strictly increasing and continuous in the interior D_I^o of its domain. Consequently, the LDP given defined by (1.2.1)-(1.2.2) and the LDP defined by (1.2.3) are equivalent.

(2). The equivalent forms (1.2.1)-(1.2.2) and (1.2.3) holds.

Proof. Let ξ be an independent random variable with distribution

$$\mathbb{P}\{\xi = -1\} = \mathbb{P}\{\xi = 1\} = \frac{1}{2}.$$

We have

$$\mathbb{E}\exp\Big\{\theta b_n \xi Y_n\Big\} = \mathbb{E}\exp\Big\{-\theta b_n Y_n\Big\} + \mathbb{E}\exp\Big\{\theta b_n Y_n\Big\} \qquad \theta \in \mathbb{R}.$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\Big\{\theta b_n \xi Y_n\Big\} = \Lambda(|\theta|) \qquad \theta \in \mathbb{R}.$$

By assumption the function $\Lambda(|\theta|)$ is essentially smooth. Applying Theorem 1.1.4 to the sequence $\{\xi Y_n\}$ we have the large deviation principle given in (1.2.1) and (1.2.2) with the rate function given by

$$I(\lambda) = \sup_{\theta\in\mathbb{R}} \Big\{\theta\lambda - \Lambda(|\theta|)\Big\} = \sup_{\theta>0} \Big\{\theta\lambda - \Lambda(\theta)\Big\}. \tag{1.2.6}$$

By Theorem 1.2.1 it remains to show that $I(\lambda)$ is restrictively increasing and continuous on $\mathbb{R}^+$. By (1.2.5), $I(\lambda)$ is non-decreasing. Consequently, there is $0 < a \le \infty$ such that $\mathcal{D}_I^o = [0, a)$. Let $\lambda \in \mathcal{D}_I^o$ be fixed. By essential smoothness of $\Lambda(|\theta|)$ and (1.2.6), the function

$$h(\theta) = \lambda\theta - \Lambda(\theta) \qquad \theta \ge 0$$

is bounded and reaches its supremum $I(\lambda)$ at some $\theta \ge 0$. Further, $\theta > 0$ for otherwise (observe that $\Lambda(0) = 0$) $\lambda\widetilde{\theta} \ge \Lambda(\widetilde{\theta})$ for all $\widetilde{\theta} > 0$, which contradicts the assumption that $\Lambda'(\widetilde{\theta}) \longrightarrow \infty$ as θ_0 approaches the right boundary of $\mathcal{D}_\Lambda^o$.

In summary, for any $\lambda \in \mathcal{D}_I^o$ there is a $\theta > 0$, such that $\Lambda'(\theta) = \lambda$ and that $I(\lambda) = \lambda\theta - \Lambda(\theta)$.

Let $0 \le \lambda_1 < \lambda_2 < a$ and find $\theta_1 > 0$ such that $I(\lambda_1) = \lambda\theta_1 - \Lambda(\theta_1)$. We have

$$I(\lambda_2) \ge \theta_1\lambda_2 - \Lambda(\theta_1) > \theta_1\lambda_1 - \Lambda(\theta_1) = I(\lambda_1).$$

Hence, $I(\cdot)$ is strictly increasing in $\mathcal{D}_I^o$.

Being increasing and lower semi-continuous, $I(\cdot)$ is left continuous. To establish continuity for (λ) in $\mathcal{D}_I^o$, therefore. all we need to show is that for any $\lambda_n, \lambda_o \in \mathcal{D}_I^o$ with $\lambda_n > \lambda_o$ and $\lambda_n \to \lambda_o^+$, $I(\lambda_n) \longrightarrow I(\lambda_o)$. Indeed, find $\theta_n > 0$ such that

$$I(\lambda_n) = \theta_n \lambda_n - \Lambda(\theta_n) \quad \text{and} \quad \Lambda'(\theta_n) = \lambda_n \qquad n = 1, 2, \cdots .$$

In particular, the sequence $\{\Lambda'(\theta_n)\}$ is bounded. By essential smoothness of $\Lambda(\cdot)$, $\{\theta_n\}$ is bounded.

Hence

$$0 \le I(\lambda_n) - I(\lambda_o) = \theta_n \lambda_n - \Lambda(\theta_n) - I(\lambda_o) \le \theta_n(\lambda_n - \lambda_o) \longrightarrow 0$$

as $n \to \infty$. □

The major step of establishing a large deviation principle is to computing exponential moment generating function

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta b_n Y_n\Big\}.$$

When the exponential moment generating function is too difficult to deal with, we may look for some other moment functions instead. For example, we may consider the large deviations under the existence of the limit

$$\Lambda_p(\theta) \equiv \lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta b_n Y_n^{1/p}\Big\} \qquad \theta > 0 \tag{1.2.7}$$

where $p > 0$ is fixed.

Corollary 1.2.4 *Assume that for all $\theta > 0$, the limit $\Lambda_p(\theta)$ given in (1.2.7) exists as an extended real number, and that $\Lambda_p(\theta) < \infty$ for some $\theta > 0$. Let $\Lambda_p(|\theta|)$ be essentially smooth in $\mathcal{D}_{\Lambda_p}^o$. Then*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \ge \lambda\} = -I_p(\lambda) \qquad \lambda > 0 \tag{1.2.8}$$

where

$$I_p(\lambda) = \sup_{\theta > 0} \Big\{\theta \lambda^{1/p} - \Lambda_p(\theta)\Big\} \qquad \lambda > 0. \tag{1.2.9}$$

Proof. Replacing Y_n by $Y_n^{1/p}$ in Theorem 1.2.3 completes the proof. □

In view of Taylor expansion,

$$\mathbb{E} \exp\Big\{\theta b_n Y_n^{1/p}\Big\} = \sum_{m=0}^{\infty} \frac{\theta^m}{m!} b_n^m \mathbb{E} Y_n^{m/p} \tag{1.2.10}$$

one may attempt to estimate $\mathbb{E}Y_n^{m/p}$ when establishing (1.2.7) by "standard" approach becomes technically difficult. When $p \neq 1$, however, it is not very pleasant to deal with the (possibly) fractional power m/p. To resolve this problem, we introduce the following lemma.

Lemma 1.2.5 *Let Ψ: $[0,\infty) \longrightarrow [0,\infty]$ be a non-decreasing lower semi-continuous function. Assume that the domain of Ψ has a form*

$$\mathcal{D}_\Psi \equiv \{\theta;\ \Psi(\theta) < \infty\} = [0, a)$$

where $0 < a \leq \infty$; and that $\Psi(\theta)$ is continuous on $\mathcal{D}_\Psi$.

Conclusion (1):

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} b_n^m \Big(\mathbb{E}Y_n^m\Big)^{1/p} \geq \Psi(\theta) \qquad \theta > 0 \tag{1.2.11}$$

if and only if

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\Big\{\theta b_n Y_n^{1/p}\Big\} \geq p\Psi\Big(\frac{\theta}{p}\Big) \qquad \theta > 0. \tag{1.2.12}$$

Conclusion (2):

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} b_n^m \Big(\mathbb{E}Y_n^m\Big)^{1/p} \leq \Psi(\theta) \qquad \theta > 0 \tag{1.2.13}$$

if and only if

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\Big\{\theta b_n Y_n^{1/p}\Big\} \leq p\Psi\Big(\frac{\theta}{p}\Big) \qquad \theta > 0. \tag{1.2.14}$$

Proof. We first prove "(1.2.11) implies (1.2.12)". We may assume that in (1.2.12), the right hand side is positive. By the expansion (1.2.10) we have

$$\mathbb{E}\exp\Big\{\theta b_n Y_n^{1/p}\Big\} \geq \frac{\theta^{[pm]+1}}{([pm]+1)!} b_n^{[pm]+1} \mathbb{E}Y_n^{\frac{[pm]+1}{p}} \qquad m = 0, 1, \cdots.$$

By Jensen inequality

$$b_n^{[pm]+1} \mathbb{E}Y_n^{\frac{[pm]+1}{p}} \geq \Big\{b_n^{pm} \mathbb{E}Y_n^m\Big\}^{\frac{[pm]+1}{pm}}.$$

Therefore, as $b_n^{pm}\mathbb{E}Y_n^m \geq 1$ we have

$$b_n^{[pm]+1} \mathbb{E}Y_n^{\frac{[pm]+1}{p}} \geq b_n^{pm} \mathbb{E}Y_n^m.$$

For any $0 < \delta < \epsilon$, by Stirling formula there is $c > 0$ such that

$$\frac{\theta^{[pm]+1}}{([pm]+1)!} \geq c^p \frac{\theta^{pm}}{(m!)^p} \Big(\frac{1+\delta}{p(1+\epsilon)}\Big)^{pm}.$$

Thus,

$$(1+\delta)^{-m}\Big(\mathbb{E}\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\Big)^{1/p} \geq c\frac{1}{m!}\Big(\frac{\theta}{p(1+\epsilon)}\Big)^m b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}$$

whenever

$$m \in Q \equiv \Big\{m; \quad b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p} \geq 1\Big\}.$$

Summing up over m,

$$\frac{1+\delta}{\delta}\Big(\mathbb{E}\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\Big)^{1/p} \geq c\sum_{m\in Q}\frac{1}{m!}\Big(\frac{\theta}{(1+\epsilon)p}\Big)^m b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}$$

Consequently,

$$\begin{aligned}&\frac{1+\delta}{\delta}\Big(\mathbb{E}\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\Big)^{1/p} + c\exp\Big\{\frac{\theta}{(1+\epsilon)p}\Big\} \\ &\geq c\sum_{m=1}^{\infty}\frac{1}{m!}\Big(\frac{\theta}{(1+\epsilon)p}\Big)^m b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}.\end{aligned} \tag{1.2.15}$$

Thus,

$$\max\Big\{0, \quad \frac{1}{p}\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\Big\} \geq \Psi\Big(\frac{\theta}{(1+\epsilon)p}\Big).$$

Letting $\epsilon \to 0^+$ on the right hand side, by the lower semi-continuity of $\Psi(\cdot)$,

$$\max\Big\{0, \quad \frac{1}{p}\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\Big\} \geq \Psi\Big(\frac{\theta}{p}\Big).$$

Since the right hand side is positive, we have proved (1.2.12).

The same estimate can be used to establish "(1.2.14) implies (1.2.13)". To proceed we may assume that the left hand side of (1.2.13) is positive.

Replacing θ by $(1+\epsilon)p\theta$ in (1.2.15) gives

$$\begin{aligned}&\frac{1+\delta}{\delta}\Big(\mathbb{E}\exp\Big\{(1+\epsilon)p\theta b_n Y_n^{1/p}\Big\}\Big)^{1/p} + ce^{\theta} \\ &\geq c\sum_{m=1}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}.\end{aligned} \tag{1.2.16}$$

By (1.2.14) (with θ being replaced by $(1+\epsilon)p\theta$)

$$\max\Big\{0,\ \Psi\big((1+\epsilon)\theta\big)\Big\} \ge \limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty \frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}.$$

Since the right hand side is positive, we must have

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty \frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p} \le \Psi\big((1+\epsilon)\theta\big).$$

To continue, we may assume that $\theta \in \mathcal{D}_\Psi$ (for otherwise (1.2.13) is trivial). By continuity of $\Psi(\cdot)$, letting $\epsilon \to 0^+$ on the right hand side gives (1.2.13).

We now prove that (1.2.12) implies (1.2.11). We may assume that the right hand side of (1.2.11) is positive. For any $k \ge 0$,

$$\frac{\theta^{[p^{-1}k]+1}}{([p^{-1}k]+1)!}b_n^{[p^{-1}k]+1}\Big[\mathbb{E}Y_n^{[p^{-1}k]+1}\Big]^{1/p} \le \sum_{m=0}^\infty \frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}.$$

By Jensen inequality, Stirling formula and an argument similar to the one used for (1.2.15), we can prove that for any $0 < \delta < \epsilon$ there is $C > 0$ independent of n such that

$$\begin{aligned}&\mathbb{E}\exp\Big\{\frac{p\theta}{1+\epsilon}b_nY_n^{1/p}\Big\} \\ &\le \exp\Big\{\frac{p\theta}{1+\epsilon}\Big\} + C\frac{1+\delta}{\delta}\Big(\sum_{m=0}^\infty \frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}\Big)^p.\end{aligned} \tag{1.2.17}$$

By (1.2.12) (with θ being replaced by $\dfrac{p\theta}{1+\epsilon}$)

$$p\Psi\Big(\frac{\theta}{1+\epsilon}\Big) \le \max\Big\{0,\ \ p\liminf_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty \frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}\Big\}.$$

Since the left hand side is positive,

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty \frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p} \ge \Psi\Big(\frac{\theta}{1+\epsilon}\Big).$$

By lower semi continuity of $\Psi(\cdot)$, letting $\epsilon \to 0^+$ on the right hand side gives (1.2.11).

Finally, "(1.2.13) implies (1.2.14)" also follows from the estimate given in (1.2.17). □

An immediate application of Lemma 1.2.5 is the following Gärtener-Ellis-type theorem.

Theorem 1.2.6 *Assume that for each $\theta > 0$, the limit*

$$\Psi(\theta) \equiv \lim_{n\to\infty} \frac{1}{b_n} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} b_n^m \Big(\mathbb{E}Y_n^m\Big)^{1/p} \tag{1.2.18}$$

exists as an extended real number. Assume that the function $\Psi(|\theta|)$ is essentially smooth. For each $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \geq \lambda\} = -I_\Psi(\lambda) \tag{1.2.19}$$

where the rate function $I_\Psi(\cdot)$ is defined by

$$I_\Psi(\lambda) = p \sup_{\theta>0} \big\{\theta\lambda^{1/p} - \Psi(\theta)\big\}. \qquad \lambda > 0. \tag{1.2.20}$$

Proof. By Lemma 1.2.5, the condition posed in Corollary 1.2.4 is satisfied with

$$\Lambda_p(\theta) = p\Psi\Big(\frac{\theta}{p}\Big) \qquad \theta > 0.$$

□

We now consider the case of a single random variable.

Theorem 1.2.7 *Let $Y \geq 0$ be a random variable such that*

$$\lim_{m\to\infty} \frac{1}{m} \log \frac{1}{(m!)^\gamma} \mathbb{E}Y^m = -\kappa \tag{1.2.21}$$

for some $\gamma > 0$ and $\kappa \in \mathbb{R}$. Then

$$\lim_{t\to\infty} t^{-1/\gamma} \log \mathbb{P}\{Y \geq t\} = -\gamma e^{\kappa/\gamma}. \tag{1.2.22}$$

Proof. We check the condition posed in Theorem 1.2.5 with $Y_t = Y/t$, $b_t = t^{1/\gamma}$ and $p = 2\gamma$. Indeed, for any $\theta > 0$,

$$\begin{aligned} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} t^{m/(2\gamma)} \Big(\mathbb{E}Y^m\Big)^{\frac{1}{2\gamma}} &\sim \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} t^{m/(2\gamma)} \Big((m!)^\gamma e^{-\kappa m}\Big)^{\frac{1}{2\gamma}} \\ &= \log \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \Big(\theta t^{\frac{1}{2\gamma}} e^{-\frac{\kappa}{2\gamma}}\Big)^m. \end{aligned}$$

By the decomposition

$$\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\Big(\theta t^{\frac{1}{2\gamma}}e^{-\frac{\kappa}{2\gamma}}\Big)^m$$
$$=\sum_{m=0}^{\infty}\frac{1}{\sqrt{(2m)!}}\Big(\theta t^{\frac{1}{2\gamma}}e^{-\frac{\kappa}{2\gamma}}\Big)^{2m}+\sum_{m=0}^{\infty}\frac{1}{\sqrt{(2m+1)!}}\Big(\theta t^{\frac{1}{2\gamma}}e^{-\frac{\kappa}{2\gamma}}\Big)^{2m+1}.$$

By Stirling formula,

$$(2m)! = \big(1+o(1)\big)^m(2^m m!)^2 \quad\text{and}\quad (2m+1)! = \big(1+o(1)\big)^m(2^m m!)^2$$

as $m\to\infty$.

Thus,

$$\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\Big(\theta t^{\frac{1}{2\gamma}}e^{-\frac{\kappa}{2\gamma}}\Big)^m \sim \log\sum_{m=0}^{\infty}\frac{1}{2^m m!}\Big(\theta t^{\frac{1}{2\gamma}}e^{-\frac{\kappa}{2\gamma}}\Big)^{2m} = \frac{1}{2}\theta^2 t^{1/\gamma}e^{-\kappa/\gamma}$$

Summarizing our discussion,

$$\lim_{to\infty} t^{-1/\gamma}\log\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}t^{m/(2\gamma)}\Big(\mathbb{E}Y^m\Big)^{\frac{1}{2\gamma}} = \frac{1}{2}\theta^2 e^{-\kappa/\gamma}.$$

Therefore, (1.2.22) follows from Theorem 1.2.5 and the fact that

$$I_\Psi(\lambda) = 2\gamma\sup_{\theta>0}\Big\{\theta\lambda^{\frac{1}{2\gamma}}-\frac{1}{2}\theta^2e^{-\kappa/\gamma}\Big\} = \lambda^{\frac{1}{2\gamma}}\gamma e^{\kappa/\gamma} \qquad (\theta>0).$$

(so $I_\Psi(1)=\gamma e^{\kappa/\gamma}$ appearing on the right hand side of (1.2.22)). $\square$

The following theorem appears as an inverse to Theorem 1.2.6.

Theorem 1.2.8 *Let $I(\lambda)$ be a non-decreasing rate function $I(\lambda)$ on $\mathbb{R}^+$ with $I(0)=0$.*

(1). Assume that

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\{Y_n\ge\lambda\}\ge -I(\lambda) \qquad (\lambda>0). \tag{1.2.23}$$

Then for every $\theta>0$

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}\ge\sup_{\lambda>0}\Big\{\theta\lambda^{1/p}-p^{-1}I(\lambda)\Big\}. \tag{1.2.24}$$

(2). Assume that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\{Y_n\ge\lambda\}\le -I(\lambda)\qquad(\lambda>0).\tag{1.2.25}$$

Then for every $\theta>0$ satisfying

$$\lim_{l\to\infty}\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}\big(Y_n^m 1_{\{Y_n\ge l\}}\big)\Big)^{1/p}=0.\tag{1.2.26}$$

we have

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}\le\sup_{\lambda>0}\Big\{\theta\lambda^{1/p}-p^{-1}I(\lambda)\Big\}.\tag{1.2.27}$$

Proof. For any $\lambda>0$,

$$\mathbb{E}Y_n^m\ge\lambda^m\mathbb{P}\{Y_n\ge\lambda\}.$$

Consequently,

$$\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}\ge\exp\big\{b_n\theta\lambda^{1/p}\big\}\Big(\mathbb{P}\{Y_n\ge\lambda\}\Big)^{1/p}.$$

By (1.2.23)

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}\ge\theta\lambda^{1/p}-p^{-1}I(\lambda).$$

Taking supremum over $\lambda>0$ on the right hand side give (1.2.24).

To prove (1.2.27), write

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}\\&=\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}\big(Y_n^m 1_{\{Y_n<l\}}\big)\Big)^{1/p}\\&+\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}\big(Y_n^m 1_{\{Y_n\ge l\}}\big)\Big)^{1/p}.\end{aligned}\tag{1.2.28}$$

Given $\epsilon>0$, partition the interval $[0,l]$ into $0=\lambda_0<\cdots<\lambda_N=l$ such that for each $1\le i\le l$ the length of the sub-interval $A_i=[\lambda_{i-1},\lambda_i]$ is less than ϵ. So we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}\big(Y_n^m 1_{\{Y_n<l\}}\big)\Big)^{1/p}\\
&\le\sum_{i=1}^{N}\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\lambda_i^{m/p}\Big(\mathbb{P}\{Y_n\in A_i\}\Big)^{1/p}\\
&\le\sum_{i=1}^{N}\exp\big\{\theta b_n(\lambda_{i-1}+\epsilon)\big\}\Big(\mathbb{P}\{Y_n\ge\lambda_{i-1}\}\Big)^{1/p}.
\end{aligned}$$

Consequently, by (1.2.25)

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}\big(Y_n^m 1_{\{Y_n<l\}}\big)\Big)^{1/p}\\
&\le\theta\epsilon+\max_{1\le i\le N}\Big\{\theta\lambda_{i-1}^{1/p}-p^{-1}I(\lambda_{i-1})\Big\}\\
&\le\theta\epsilon+\sup_{\lambda>0}\Big\{\theta\lambda^{1/p}-p^{-1}I(\lambda)\Big\}.
\end{aligned}$$

Letting $\epsilon\to 0^+$ on the right hand side,

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}\big(Y_n^m 1_{\{Y_n<l\}}\big)\Big)^{1/p}\le\sup_{\lambda>0}\Big\{\theta\lambda^{1/p}-p^{-1}I(\lambda)\Big\}.$$

By the decomposition (1.2.28),

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}Y_n^m\Big)^{1/p}\\
&\le\max\bigg\{\sup_{\lambda>0}\Big\{\theta\lambda^{1/p}-p^{-1}I(\lambda)\Big\},\\
&\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}\big(Y_n^m 1_{\{Y_n\ge l\}}\big)\Big)^{1/p}\bigg\}.
\end{aligned}$$

By (1.2.25), letting $l\to\infty$ on the right hand side leads to (1.2.26). □

The condition (1.2.26) is called *exponential uniform integrability.* It can be examined through the following lemma.

Lemma 1.2.9 *Given $\theta>0$, the assumption (1.2.26) holds if there is $\epsilon>0$ such that either one of the following happens:*

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{p(1+\epsilon)\theta b_n Y_n^{1/p}\Big\}<\infty. \tag{1.2.29}$$

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{m=0}^{\infty} \frac{(1+\epsilon)^m \theta^m}{m!} b_n^m \Big(\mathbb{E}Y_n^m\Big)^{1/p} < \infty. \tag{1.2.30}$$

Proof. The conclusion follows from the second part of Lemma 1.2.5 and a standard application of Chebyshev inequality. □

1.3 LDP by sub-additivity

A random sequence $\{Y_n\}$ is said to be *sub-additive* , if for any $n, m \geq 1$,

$$Y_{n+m} \leq Y_n + Y'_m$$

where $Y_m \overset{d}{=} Y_m$ and Y'_m is independent of $\{Y_1, \cdots, Y_n\}$.

In particular, a *deterministic sub-additive sequence* $\{a(n)\}$ is defined by the inequality $a(n+m) \leq a(n) + a(m)$.

Lemma 1.3.1 *For any deterministic sub-additive sequence* $\{a(n)\}$*, the equality*

$$\lim_{n\to\infty} n^{-1}a(n) = \inf_{m\geq 1} m^{-1}a(m)$$

holds in the extended real line $[-\infty, \infty)$*.*

Proof. All we need is to show that

$$\limsup_{n\to\infty} n^{-1}a(n) \leq \inf_{m\geq 1} m^{-1}a(m). \tag{1.3.1}$$

Let $m \geq 1$ be fixed but arbitrary. For any big n, write $n = km + r$, where $k \geq 1$ and $0 \leq r < m$ are integers. By sub-additivity,

$$a(n) \leq ka(m) + a(r).$$

Consequently,

$$\limsup_{n\to\infty} n^{-1}a(n) \leq m^{-1}a(m)$$

Taking inframum over m on the right hand side leads to (1.3.1). □

Given a sub-additive random sequence $\{Y_n\}_{n\geq 1}$ and $\theta > 0$, we have that

$$\mathbb{E}\exp\big\{\theta Y_{n+m}\big\} \leq \mathbb{E}\exp\big\{\theta Y_n\big\}\mathbb{E}\exp\big\{\theta Y_m\big\}.$$

Consequently,

$$a(n) \equiv \log \mathbb{E} \exp\{\theta Y_n\} \qquad n = 1, 2, \cdots$$

is a deterministic sub-additive sequence. Consequently, we have the following corollary.

Corollary 1.3.2 *Let $\{Y_n\}_{n\geq 1}$ be sub-additive. For any $\theta > 0$, the equality*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp\{\theta Y_n\} = \inf_{m\geq 1} \frac{1}{m} \log \mathbb{E} \exp\{\theta Y_m\}$$

holds in the extended real line $[-\infty, \infty]$.

In the following theorem, we establish exponential integrability for Y_n.

Theorem 1.3.3 *Let $\{Y_n\}$ be a sub-additive random sequence such that $Y_1 \leq C$ a.s. for some constant $C > 0$. Let $c_n \geq 1$ be a deterministic sequence such that the normalized sequence*

$$\max_{k\leq n} Y_k / c_n \qquad n = 1, 2, \cdots$$

is stochastically bounded. Then there is a $\theta > 0$ such that

$$\sup_n \mathbb{E} \exp\Big\{\theta Y_n / c_n\Big\} < \infty. \tag{1.3.2}$$

Further, if

$$\liminf_{n\to\infty} c_{mn}/c_n > 1 \tag{1.3.3}$$

for some $m > 1$, then (1.3.2) holds for all $\theta > 0$.

Proof. We first show that for any $s, t > 0$, and $0 < \lambda < 1$,

$$\sum_{k=1}^{\infty} \lambda^k \mathbb{P}\Big\{Y_k \geq s + t + C\Big\} \leq \mathbb{E}\lambda^{\mathcal{T}_s} \sum_{k=1}^{\infty} \lambda^k \mathbb{P}\Big\{Y_k \geq t\Big\} \tag{1.3.4}$$

where for each $s > 0$,

$$\mathcal{T}_s = \inf\Big\{j \geq 1; \quad Y_j \geq s\Big\}.$$

Notice that for any $j \geq 1$, by sub-additivity

$$Y_j - Y_{j-1} \leq Y_1' \stackrel{d}{=} Y_1 \leq C \qquad a.s.$$

For any $k \ge 1$,

$$\mathbb{P}\Big\{Y_k \ge s+t+C\Big\} = \mathbb{P}\Big\{\mathcal{T}_s \le k-1;\ Y_k \ge s+t+C\Big\}$$
$$\le \sum_{j=1}^{k-1}\mathbb{P}\Big\{\mathcal{T}_s = j,\ Y_n - Y_{j-1} \ge t+C\Big\} \le \sum_{j=1}^{k-1}\mathbb{P}\Big\{\mathcal{T}_s = j,\ Y_k - Y_j \ge t\Big\}$$

where the second step follows from the fact that $Y_{\mathcal{T}_s - 1} \le s$. By sub-additivity, $Y_k - Y_j \le Y'_{k-j} \stackrel{d}{=} Y_{k-j}$ and Y'_{k-j} is independent of $\{\tau = j\}$. Therefore,

$$\mathbb{P}\Big\{Y_k \ge s+t+C\Big\} \le \sum_{j=1}^{k-1}\mathbb{P}\big\{\mathcal{T}_s = j\big\}\mathbb{P}\big\{Y_{k-j} \ge t\big\}.$$

Multiplying λ^k on the both sides and summing up over k lead to (1.3.4).

Given $N > 0$ and integers $m, n \ge 1$, by (1.3.4),

$$\sum_{k=1}^{\infty}\lambda^k\mathbb{P}\Big\{Y_k \ge m(Nc_n + C)\Big\} \le \Big(\mathbb{E}\lambda^{\mathcal{T}_{Nc_n}}\Big)^m.$$

Taking $\lambda = \exp\{-n^{-1}\}$ we have that

$$\sum_{k=1}^{\infty}\exp\{-n^{-1}k\}\mathbb{P}\Big\{Y_k \ge m(Nc_n + C)\Big\} \le \Big(\mathbb{E}\exp\big\{-\mathcal{T}_{Nc_n}/n\big\}\Big)^m.$$

By assumption, for any $M > 0$,

$$\lim_{N\to\infty}\sup_{n\ge 1}\mathbb{P}\big\{\mathcal{T}_{Nc_n} \le Mn\big\} = 0.$$

In particular, one can make $N > 0$ sufficiently large, so that

$$\mathbb{E}\exp\big\{-\mathcal{T}_{Nc_n}/n\big\} \le e^{-2}.$$

Hence, uniformly on $n \ge 1$,

$$\mathbb{P}\Big\{Y_n \ge m(Nc_n + C)\Big\} \le ee^{-2m} \qquad m = 1, 2, \cdots.$$

Consequently, (1.3.2) holds for some $\theta > 0$.

Assuming (1.3.3), we now show that (1.3.2) holds for every $\theta > 0$. Let $\theta_0 > 0$ satisfy (1.3.2). By (1.3.3) there is $m > 1$ such that

$$c_{[n/m]}/c_n \le \theta_0/\theta$$

for all $n \ge 1$. By sub-additivity we have that

$$\mathbb{E}\exp\left\{\theta c_n^{-1} Y_n\right\} \le \left(\mathbb{E}\exp\left\{\theta c_n^{-1} Y_{[n/m]}\right\}\right)^m \le \left(\mathbb{E}\exp\left\{\theta_0 c_{[n/m]}^{-1} Y_{[n/m]}\right\}\right)^m.$$

The right hand side is bounded uniformly over $n \ge 1$. $\square$

The notion of sub-additivity can be extended to the stochastic processes with continuous time. A stochastic process Z_t $(t \ge 0)$ is said to be *sub-additive*, if for any $s, t \ge 0$, $Z_{s+t} \le Z_s + \widetilde{Z}_t$ for a random variable $\widetilde{Z}_t$ independent of $\{Z_u;\ \ 0 \le u \le s\}$ with $\widetilde{Z}_t \overset{d}{=} Z_t$. With completely parallel argument we have

Lemma 1.3.4 *For any deterministic sub-additive function $\{a(t)\}$ $(t \in \mathbb{R}^+)$, the equality*

$$\lim_{t\to\infty} n^{-1} a(t) = \inf_{s>0} s^{-1} a(s)$$

holds in the extended real line $[-\infty, \infty)$.

We may restate Theorem 1.3.3 in the setting of continuous time. Rather, we give the following slightly different version.

Theorem 1.3.5 *For any non-decreasing sub-additive process Z_t with continuous path and with $Z_0 = 0$,*

$$\mathbb{E}\exp\{\theta Z_1\} < \infty \tag{1.3.5}$$

for any $\theta > 0$. Consequently,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{\theta Z_t\right\} = \Psi(\theta) \tag{1.3.6}$$

exists with $0 \le \Psi(\theta) < \infty$ for every $\theta > 0$.

Proof. Clearly, we need only to establish (1.3.5). By sample path continuity and by monotonicity, the continuous version of (1.3.4)

$$\mathbb{P}\left\{Z_t \ge a + b\right\} \le \mathbb{P}\left\{Z_t \ge a\right\}\mathbb{P}\left\{Z_t \ge b\right\} \tag{1.3.7}$$

holds for any $a, b > 0$.

Consequently,

$$\mathbb{P}\left\{Z_t \ge ma\right\} \le \left(\mathbb{P}\left\{Z_t \ge a\right\}\right)^m \qquad m = 1, 2, \cdots.$$

For any $a > 0$, by the fact $Z_0 = 0$ and by sample path continuity, one can have $\mathbb{P}\{Z_t \ge a\} \le e^{-2}$ by making $t > 0$ sufficiently small. Consequently, for any $\theta > 0$,

$$\mathbb{E}\exp\{\theta Z_t\} < \infty$$

for small t. We may take $t < 1$. By sub-additivity

$$\mathbb{E}\exp\{\theta Z_1\} \le \Big(\mathbb{E}\exp\{\theta Z_t\}\Big)^{[t^{-1}]+1} < \infty.$$

$\square$

1.4 Notes and comments

Section 1.1.

The earliest recorded work in large deviation theory is due to Cramér ([37]) and was published in 1938. The literature on Large deviations is massive and it is impossible to list even small portion of it. We point the fundamental roles played by Donsker and Varadhan; Freidlin and Wentzell in the birth of modern theory of large deviations. The idea that the limit of the logarithmic moment generating function decides the large deviation goes back to Cramér ([37]). It has been formulated by Gärtner ([80]) and Ellis ([71]) into a general theorem later known as Gi̇̈tner-Ellis theorem (Theorem 1.1.4). There are many excellent book accounts available in the theory of Large deviations. We mention here the books by Varadhan [152], Freidlin and Wentzell [76], Ellis [72], Stroock [148], Deuschel and Stroock [50], Bucklew [18], Dembo and Zeitouni [44], den Hollander [47], Feng and Kurtz [73]. Finally, we refer an interested reader to the recent survey by Varadhan [154] for an overview on the latest development in the area of large deviations.

Most of the material in this section comes from the book by Dembo and Zeitouni ([44]).

For the large deviations in infinite dimensional space, a challenging part is to establish the exponential tightness. Theorem 1.1.7 provides a practical way of examining the exponential tightness. This useful result is due to de Acosta ([41]).

Exercise 1.

In an infinite Bernoulli trial with the success rate $0 < \theta < 1$, let B_n be the number of successes in the first n games. Recall that

$$\lim_{n\to\infty} \frac{1}{n} B_n = \theta \qquad a.s.$$

Apply Gärtner-Ellis theorem to compute the limit

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\Big|\frac{B_n}{n}-\theta\Big|\ge\epsilon\Big\}$$

where $0<\epsilon<\min\{1-\theta,\ \theta\}$.

Section 1.2.

Much material in this section existing in some recent research papers instead of standard textbooks. Theorem 1.2.2 was essentially obtained in the paper by Bass, Chen and Rosen ([7]), Theorem 1.2.6 appeared in Chen ([24], [25]). Theorem 1.2.7 is due to König and Mörters ([107]).

Section 1.3.

Argument by sub-additivity has becomes a sophisticated tool in the general frame work of large deviations. Very often in literature, it is the deterministic sub-additivity formulated in Lemma 1.3.1 that is used to prove the existence of logarithmic moment generating function.

Exercise 2. Prove (1.3.7).

Chapter 2

Brownian intersection local times

–

2.1 Introduction

Recall that a d-dimensional *Brownian motion* $W(t)$ is defined as a stochastic process in $\mathbb{R}^d$ with the following properties:

(1). For any $t > 0$, $W(t)$ is a normal random variable with mean $\mathbf{0}$ and covariance matrix $t\mathbf{I}_d$, where $\mathbf{I}_d$ is the $d \times d$ identity matrix.

(2). For any $s < t$, the increment $W(t) - W(s)$ is independent of the family

$$\Big\{W(u); \quad u \le s\Big\}$$

and has a distribution same as $W(t-s)$.

We follow the convention that $W(0) = 0$ most of the time, with the exception when $W(t)$ is viewed as a Markov process. Brownian motion is also known as Wiener process. Due to its importance, Brownian motion has been studied extensively. It is virtually impossible, in the scope of this book, to list even the major results in literature about Brownian motion. We refer the reader to the books of Ito and McKean [93], Revuz and Yor [133], Rogers and Williams [136], [137] .

The notion of *intersection local times* have been introduced to measure the intensity of the Brownian path intersections. For a d-dimensional Brownian motion $W(t)$, an integer $p \geq 2$ and a p-multiple time set

$$A \subset \big\{(t_1, \cdots, t_p) \in (\mathbb{R}^+)^p; \quad t_1 < \cdots < t_p\big\}$$

the random quantity $\beta(A)$ formally written as

$$\beta(A) = \int_A \prod_{j=2}^p \delta_0\big(W(s_j) - W(s_{j-1})\big) ds_1 \cdots ds_p \tag{2.1.1}$$

measures the amount of p-multiple self-intersections over the time set A. The random measure $\beta(A)$ is called (p-multiple) self-intersection local time of the Brownian motion $W(t)$.

Let $W_1(t), \cdots, W_p(t)$ be independent d-dimensional Brownian motions. For any $A \subset (\mathbb{R}^+)^p$, the random quantity formally written as

$$\alpha(A) = \int_A \prod_{j=2}^p \delta_0\big(W_{j-1}(s_{j-1}) - W_j(s_j)\big) ds_1 \cdots ds_p \tag{2.1.2}$$

measures the amount of mutual intersection over the time set A. In literature, $\alpha\big([0, t_1] \times \cdots \times [0, t_p]\big)$ is called (p-multiple) mutual intersection local time of the Brownian motions $W_1(t), \cdots, W_p(t)$.

In addition to its mathematical importance, the study of intersection local times is also motivated by the needs from physics. Physicists concern about geometric shape of the polymer which is often described by a suitable random path (such as a Brownian curve). The geometry of a polymer is decided by the intensity that the random path intersects itself. Write $[0, t]_<^2 = \{(r, s); \quad 0 \leq r \leq s \leq t\}$. Then the quantity

$$\beta\big([0, t]_<^2\big) = \iint_{\{0 \leq r < s \leq t\}} \delta_0\big(W(r) - W(s)\big) dr ds$$

measures (double) self-intersection up to time t. The case when $\beta\big([0, t]_<^2\big)$ is large corresponds to a "contracting" polymer; while the case when $\beta\big([0, t]_<^2\big)$ is small corresponds to a "spread-out" polymer. The geometric shape of a polymer is often influenced by the environment (media). If the environment encourages attraction among the molecules, then the polymer is contracting. In this case, the polymer is called self-attracting polymer. In the opposite case, the polymer is spread-out and is called self-repelling polymer.

In physics, the probability measure $\widehat{\mathbb{P}}_\lambda$ on $C\big\{[0, t], \mathbb{R}^2\big\}$ defined as

$$\widehat{\mathbb{P}}_\lambda(B) = \widehat{C}_\lambda^{-1} \mathbb{E}\bigg(\exp\Big\{\lambda \beta\big([0, t]_<^2\big)\Big\} 1_{\{W(\cdot) \in B\}}\bigg), \tag{2.1.3}$$

where $B \in C\{[0,t],\mathbb{R}^2\}$ is Borel measurable, is regarded as the distribution of the curve representing a self-attracting planar polymer, where $\lambda > 0$ represents certain media parameter (such as temperature) and $\widehat{C}_\lambda$ is the normalizing constant making $\widehat{\mathbb{P}}_\lambda$ a probability measure. Indeed, the curves with higher self-intersection are given more distributional weight and the degree of this favoritism is decided by the value of λ.

Similarly, a self-repelling planar polymer is modeled by the distribution

$$\widetilde{\mathbb{P}}_\lambda(B) = \widetilde{C}_\lambda^{-1}\mathbb{E}\bigg(\exp\Big\{-\lambda\beta\big([0,t]^2_<\big)\Big\}1_{\{W(\cdot)\in B\}}\bigg). \tag{2.1.4}$$

An interested reader is reffered to an excellent survey paper by van der Hofstad and König ([89]) for a systematic account on link between polymers and sample path intersections.

Intersection local times of Brownian motions needs to be properly defined. For example, we shall find out that (Proposition 2.3.6) the quantity $\beta\big([0,t]^2_<\big)$ used in (2.1.3) and (2.1.4) explodes and we shall fix this problem in Section 2.4. In addition, some basic but important properties of intersection local times, such as integrability and Le Gall's moment formula, needs to be installed. We devote this chapter to carry out these goals.

2.2 Mutual intersection local time

Prior to the construction of intersection local time $\alpha(A)$ (formally given in (2.1.2)), we need to see when the independent d-dimensional Brownian motions $W_1(t),\cdots,W_p(t)$ intersect. This problem was completely solved in 1950s by Dvoretzsky, Erdös and Kakutani ([64] and [65]). We introduce their result without proof.

Theorem 2.2.1 *The set of intersection*

$$\bigcap_{j=1}^p \{x\in\mathbb{R}^d;\quad x = W_j(t)\ \textit{for some}\ \ t\ge 0\}$$

contains points different from 0 if and only if $p(d-2) < d$.

In the following discussion we restrict to the case $d(p-2) < d$, which contains three sub-cases:

1. $d=1$, $p=2,3,\cdots$.
2. $d=2$, $p=2,3,\cdots$.

3. $d = 3$, $p = 2$.

A function playing an important role in deciding whether Brownian paths intersect is the *Green's function* $G(x)$ defined as

$$G(x) = \int_0^\infty e^{-t} p_t(x) dt \qquad x \in \mathbb{R}^d \tag{2.2.1}$$

where

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t} \qquad x \in \mathbb{R}^d \tag{2.2.2}$$

is the density function of the random variable $W(t)$. A crucial fact (Theorem A.1, Appendix) is that

$$\int_{\mathbb{R}^d} G^p(x) dx < \infty \tag{2.2.3}$$

under $p(d-2) < d$.

Given $\epsilon > 0$, define a random measure $\alpha_\epsilon(\cdot)$ on $(\mathbb{R}^+)^p$ as follows

$$\alpha_\epsilon(A) = \int_{\mathbb{R}^d} \left[\int_A ds_1 \cdots ds_p \prod_{j=1}^p p_\epsilon\big(W_j(s_j) - x\big) \right] dx \qquad A \subset (\mathbb{R}^+)^p.$$

Roughly speaking, we shall show that as $\epsilon \to 0^+$, $\alpha_\epsilon(\cdot)$ converges on all bounded p-dimensional boxes and we then show that the limiting family can be extended into a random measure which is later called the mutual intersection local time generated by the Brownian motions $W_1(t), \cdots, W_p(t)$.

For two functions f and g on $\mathbb{R}^d$, the *convolution* $f * g$ of f and g is defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x-y) dy = \int_{\mathbb{R}^d} f(x-y) g(y) dy \qquad x \in \mathbb{R}^d$$

whenever the involved integrals are well defined. For any probability density $h(x)$ on $\mathbb{R}^d$ and any $\epsilon > 0$, it is easy to see that the function

$$h_\epsilon(x) = \epsilon^{-d} h(\epsilon^{-1} x)$$

is a probability density on $\mathbb{R}^d$.

Lemma 2.2.2 *For any function $f \in \mathcal{L}^p(\mathbb{R}^d)$ with $p \geq 1$, and for any probability density $h(x)$ on $\mathbb{R}^d$,*

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^d} |f(x) - f * h_\epsilon(x)|^p dx = 0.$$

Proof. First we claim that for $\delta > 0$ there is $N > 0$ such that

$$\limsup_{\epsilon \to 0^+} \int_{\{|x| \geq N} |f * h_\epsilon(x)|^p dx < \delta \tag{2.2.4}$$

.

Indeed, by Jensen inequality,

$$\begin{aligned}
&\int_{\{|x|\geq N} |f * h_\epsilon(x)|^p dx \\
&\leq \int_{\{|x|\geq N} \int_{\mathbb{R}^d} h_\epsilon(y) |f(x-y)|^p dy dx \\
&= \int_{\mathbb{R}^d} h_\epsilon(y) \left[\int_{\{|x|\geq N} |f(x-y)|^p dx \right] dy \\
&= \int_{\{|y|\geq 1\}} h_\epsilon(y) \left[\int_{\{|x|\geq N} |f(x-y)|^p dx \right] dy \\
&+ \int_{\{|y|<1\}} h_\epsilon(y) \left[\int_{\{|x|\geq N} |f(x-y)|^p dx \right] dy.
\end{aligned}$$

For the first term on the right hand side, it is bounded by the quantity

$$\begin{aligned}
&\left\{ \int_{\mathbb{R}^d} |f(x)|^p dx \right\} \left\{ \int_{\{|y|\geq 1\}} h_\epsilon(y) dy \right\} \\
&= \left\{ \int_{\mathbb{R}^d} |f(x)|^p dx \right\} \left\{ \int_{\{|x|\geq \epsilon^{-1}\}} h(x) dx \right\}
\end{aligned}$$

that tends to zero as $\epsilon \to 0^+$.

As for the second term, it is dominated by

$$\int_{\{|x|\geq N-1\}} |f(x)|^p dx$$

which can be sufficiently small if N is sufficiently large.

In view of (2.2.4), therefore, it remains to show that for any fixed $N > 0$

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^d} |f(x) - f * h_\epsilon(x)|^p dx = 0. \tag{2.2.5}$$

The validity of this claim would be immediate if f is uniformly continuous. When this is not the case, for any given $\delta > 0$ one can pick up an uniformly continuous function $\bar{f}$ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |f(x) - \bar{f}(x)|^p dx < \delta.$$

By Jensen inequality, one can show that

$$\int_{\mathbb{R}^d} |(f * h_\epsilon)(x) - (\bar{f} * h_\epsilon)(x)|^p dx \le \int_{\mathbb{R}^d} |f(x) - \bar{f}(x)|^p dx < \delta.$$

Thus, the desired (2.2.5) follows from the fact that it is satisfied by $\bar{f}$. □

For any $A \subset \mathbb{R}^+$ and any integer $m \ge 2$, set

$$A^m_< = \big\{(s_1, \cdots, s_m) \in A^m; \quad s_1 < \cdots < s_m\big\}.$$

Theorem 2.2.3 *For any bounded Borel sets $A_1, \cdots, A_p \subset \mathbb{R}^+$, the limit*

$$\lim_{\epsilon \to 0^+} \alpha_\epsilon(A_1 \times \cdots \times A_p) = \alpha(A_1 \times \cdots \times A_p) \tag{2.2.6}$$

exists in the $\mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ for all $m > 0$. Further, for any integer $m \ge 1$,

$$\begin{aligned} &\mathbb{E}\Big[\alpha\big(A_1 \times \cdots \times A_p\big)^m\Big] \qquad (2.2.7) \\ &= \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \prod_{j=1}^p \sum_{\sigma \in \Sigma_m} \int_{(A_j)^m_<} ds_1 \cdots ds_m \\ &\qquad \times \prod_{k=1}^m p_{s_k - s_{k-1}}\big(x_{\sigma(k)} - x_{\sigma(k-1)}\big) \end{aligned}$$

where Σ_m be the permutation group over $\{1, \cdots, m\}$ and we adopt the convention that $s_0 = 0$ and $x_{\sigma(0)} = 0$.

Proof. The proof consists of three steps. First we show that the right hand side of (2.2.7) is finite. Set, for given Borel set $t \in \mathbb{R}^+$,

$$V_t = \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \Big[\sum_{\sigma \in \Sigma_m} \int_{[0,t]^m_<} ds_1 \cdots ds_m \prod_{k=1}^m p_{s_k - s_{k-1}}\big(x_{\sigma(k)} - x_{\sigma(k-1)}\big)\Big]^p.$$

By Hölder inequality and by the fact that A_j is bounded, we need only to prove that $V_t < \infty$ for any $t > 0$.

Indeed, for any $\sigma \in \Sigma_m$,

$$\begin{aligned} &\int_{[0,t]^m_<} ds_1 \cdots ds_m \prod_{k=1}^m p_{s_k - s_{k-1}}\big(x_{\sigma(k)} - x_{\sigma(k-1)}\big) \\ &\le \prod_{k=1}^m \int_0^t p_s(x_{\sigma(k)} - x_{\sigma(k-1)}) ds \\ &= \prod_{k=1}^m g_t\big(x_{\sigma(k)} - x_{\sigma(k-1)}\big) \qquad \text{(say).} \end{aligned}$$

By Jensen inequality, therefore,

$$V_t \le (m!)^{p-1} \sum_{\sigma\in\Sigma_m} \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \Big[\prod_{k=1}^m g_t(x_{\sigma(k)} - x_{\sigma(k-1)})\Big]^p \tag{2.2.8}$$
$$= (m!)^p\Big(\int_{\mathbb{R}^d} g_t^p(x)dx\Big)^m = (m!)^p\Big(\int_{\mathbb{R}^d}\Big[\int_0^t p_s(x)ds\Big]^p dx\Big)^m.$$

To proceed, we need to show that the integral on the right hand side is finite. Indeed, by By Gaussian integration,

$$\int_{\mathbb{R}^d}\Big[\int_0^t p_s(x)ds\Big]^p dx = \int_{[0,t]^p} du_1\cdots du_p \int_{\mathbb{R}^d}\prod_{j=1}^p p_{s_j}(x)dx$$
$$= (2\pi)^{-\frac{d(p-1)}{2}}\int_{[0,t]^p} du_1\cdots du_p\Big(\sum_{j=1}^p \prod_{\substack{1\le l\le p\\ l\ne j}} s_l\Big)^{-d/2}.$$

By arithmetic-geometric mean inequality,

$$\frac{1}{p}\sum_{j=1}^p \prod_{\substack{1\le l\le p\\ l\ne j}} s_l \ge \sqrt[p]{\prod_{j=1}^p \prod_{\substack{1\le l\le p\\ l\ne j}} s_l} = \prod_{j=1}^p s_j^{\frac{p-1}{p}}.$$

Therefore, by the fact that $\dfrac{d(p-1)}{2p} < 1$

$$\int_{\mathbb{R}^d}\Big[\int_0^t p_s(x)ds\Big]^p dx \le (2\pi)^{-\frac{d(p-1)}{2}}\Big\{\int_0^t s^{-\frac{d(p-1)}{2p}}ds\Big\}^p < \infty. \tag{2.2.9}$$

As the second step, we establish the identity

$$\mathbb{E}\Big[\alpha_\epsilon\big(A_1\times\cdots\times A_p\big)^m\Big] \tag{2.2.10}$$
$$= \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \prod_{j=1}^p \int_{(\mathbb{R}^d)^m} dz_1\cdots dz_m\Big(\prod_{k=1}^m p_\epsilon(z_k - x_k)\Big)$$
$$\times \sum_{\sigma\in\Sigma_m}\int_{(A_j)^m_<} ds_1\cdots ds_m \prod_{k=1}^m p_{s_k - s_{k-1}}(z_{\sigma(k)} - z_{\sigma(k-1)}).$$

By Fubini theorem,

$$\begin{aligned}
&\mathbb{E}\Big[\alpha_\epsilon\big(A_1\times\cdots\times A_p\big)^m\Big]\\
&=\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\prod_{j=1}^p\int_{A_j^m}ds_1\cdots ds_m\mathbb{E}\prod_{k=1}^m p_\epsilon\big(W(s)-x_k\big)\\
&=\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\prod_{j=1}^p\sum_{\sigma\in\Sigma_m}\int_{(A_j)^m_<}ds_1\cdots ds_m\mathbb{E}\prod_{k=1}^m p_\epsilon\big(W(s)-x_{\sigma(k)}\big).
\end{aligned}$$

For any $(s_1,\cdots,s_m)\in(A_j)^m_<$, the probability density of the random vector $\big(W(s_1),\cdots,W_p(s_m)\big)$ is

$$\prod_{k=1}^m p_{s_k-s_{k-1}}(z_k-z_{k-1})\qquad z_1,\cdots,z_m\in\mathbb{R}^d$$

where we adopt the convention that $s_0=0$ and $z_0=0$. Thus, (2.2.10) follows from the following computation.

$$\begin{aligned}
&\mathbb{E}\prod_{k=1}^m p_\epsilon\big(W(s)-x_{\sigma(k)}\big)\\
&=\int_{(\mathbb{R}^d)^m}dz_1\cdots dz_m\Big(\prod_{k=1}^m p_\epsilon(z_k-x_{\sigma(k)})\Big)\Big(\prod_{k=1}^m p_{s_k-s_{k-1}}(z_k-z_{k-1})\Big)\\
&=\int_{(\mathbb{R}^d)^m}dz_1\cdots dz_m\Big(\prod_{k=1}^m p_\epsilon(z_{\sigma(k)}-x_{\sigma(k)})\Big)\Big(\prod_{k=1}^m p_{s_k-s_{k-1}}(z_{\sigma(k)}-z_{\sigma(k-1)})\Big)\\
&=\int_{(\mathbb{R}^d)^m}dz_1\cdots dz_m\Big(\prod_{k=1}^m p_\epsilon(z_k-x_k)\Big)\Big(\prod_{k=1}^m p_{s_k-s_{k-1}}(z_{\sigma(k)}-z_{\sigma(k-1)})\Big)
\end{aligned}$$

where $z_{\sigma(0)}=0$, where the second step follows from index permutation and the third from the fact that

$$\prod_{k=1}^m p_\epsilon\big(z_{\sigma(k)}-x_{\sigma(k)}\big)=\prod_{k=1}^m p_\epsilon\big(z_k-x_k\big).$$

Before starting the third step, we reduce our task to a suitable point. For $1\le j\le p$, set

$$f_j(x_1,\cdots,x_m)=\sum_{\sigma\in\Sigma_m}\int_{(A_j)^m_<}ds_1\cdots ds_m\prod_{k=1}^m p_{s_k-s_{k-1}}(x_{\sigma(k)}-x_{\sigma(k-1)})$$

and

$$f_{\epsilon,j} = \int_{(\mathbb{R}^d)^m} dz_1 \cdots dz_m \Big(\prod_{k=1}^m p_\epsilon(z_k - x_k) \Big) f_j(z_1, \cdots, z_m).$$

By (2.2.8) $f_j \in \mathcal{L}^p(\mathbb{R}^{dm})$. By Lemma 2.2.2, $f_{\epsilon,j}$ converges to f_j in $\mathcal{L}^p(\mathbb{R}^{dm})$ as $\epsilon \to 0^+$. Observe that the multilinear form

$$L(g_1, \cdots, g_p) = \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \prod_{j=1}^p g_j(x_1, \cdots x_m) \qquad g_j \in \mathcal{L}^p(\mathbb{R}^{dm})$$

is continuous on the product space $\bigotimes_{j=1}^p \mathcal{L}^p(\mathbb{R}^{dm})$. Thus, the right hand side of (2.2.10) converges to the right hand side of (2.2.7). Therefore, it remains to show that (2.2.6) holds in the sense of $\mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ for all integers $m \geq 1$. By (2.2.10) and the convergence of the right hand side, we have

$$\sup_{\epsilon > 0} \mathbb{E}\Big[\alpha_\epsilon(A_1 \times \cdots \times A_p)^m\Big] < \infty \qquad m = 1, 2, \cdots.$$

Therefore, all we need to prove is the convergence (2.2.6) in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$. To this end, all we need is to prove that the limit

$$\lim_{\epsilon, \epsilon' \to 0^+} \mathbb{E}\Big\{\alpha_\epsilon(A_1 \times \cdots \times A_p) \cdot \alpha_{\epsilon'}(A_1 \times \cdots \times A_p)\Big\}$$

exists. That is the task of the third step.

A slight modification of the computation for (2.2.10) (with $m = 2$) gives that

$$\begin{aligned}
&\mathbb{E}\Big\{\alpha_\epsilon(A_1 \times \cdots \times A_p) \cdot \alpha_{\epsilon'}(A_1 \times \cdots \times A_p)\Big\} \\
&= \iint_{\mathbb{R}^d \times \mathbb{R}^d} dxdy \prod_{j=1}^p \iint_{\mathbb{R}^d \times \mathbb{R}^d} dz_1 dz_2 p_\epsilon(z_1 - x) p_{\epsilon'}(z_2 - y) f_j(z_1, z_2)
\end{aligned}$$

where

$$f_j(x, y) = \int_{(A_j)^2_<} dsdt \Big\{p_s(x) p_{t-s}(y - x) + p_s(y) p_{t-s}(x - y)\Big\}.$$

A slight modification of Lemma 2.2.2 gives that for each $1 \leq j \leq p$,

$$\begin{aligned}
&\iint_{\mathbb{R}^d \times \mathbb{R}^d} dxdy \bigg| f_j(x, y) \\
&- \iint_{\mathbb{R}^d \times \mathbb{R}^d} dz_1 dz_2 p_\epsilon(z_1 - x) p_{\epsilon'}(z_2 - y) f_j(z_1, z_2)\bigg|^p \\
&\longrightarrow 0 \qquad (\epsilon, \epsilon' \to 0^+).
\end{aligned}$$

This leads to the requested conclusion. $\square$

The equality (2.2.7) is called *Le Gall's moment identity* . By Hölder inequality, an immediate consequence of Le Gall's moment identity is the following inequality,

$$\mathbb{E}\Big[\alpha(A_1\times\cdots\times A_p)^m\Big]\le\prod_{j=1}^p\Big\{\mathbb{E}\Big[\alpha(A_j^p)^m\Big]\Big\}^{1/p}\qquad m=1,2,\cdots. \quad (2.2.11)$$

The next thing we intend to do is to extend the limiting family

$$\Big\{\alpha(A_1\times\cdots\times A_p);\quad \text{bounded } A_1,\cdots,A_p\subset\mathbb{R}^+\Big\}$$

into a random measure on $(\mathbb{R}^+)^p$. Strictly speaking, for each $(A_1,\cdots,A_p)$ the notation $\alpha(A_1\times\cdots\times A_p)$ represents an equivalent class of random variables which are equal to each other with probability 1. We need to prove that this family has an extendable version. Our approach is to show the random measure can be generated by a continuous distribution function. Our major tool is Le Gall's moment identity.

Here we introduce the notation

$$\mathbf{t}=(t_1,\cdots,t_p)\qquad t_1,\cdots,t_p\in\mathbb{R}^+$$

and for two multiple times parameters $\mathbf{s},\mathbf{t}\in(\mathbb{R}^+)^p$, write

$$||\mathbf{t}-\mathbf{s}||=\sqrt{(t_1-s_1)^2+\cdots+(t_p-s_p)^2}$$

and $[\mathbf{s},\mathbf{t}]=[s_1,t_1]\times\cdots\times[s_p,t_p]$ if $\mathbf{s}\le\mathbf{t}$ (which means $s_j\le t_j$ for $j=1,\cdots,p$).

Lemma 2.2.4 *Let $T>0$ be fixed. There is a constant $C=C(T)>0$ such that for any $\mathbf{s},\mathbf{t}\in[0,T]^p$ and for any $m=1,2,\cdots$,*

$$\mathbb{E}\Big|\alpha([\mathbf{0},\mathbf{t}])-\alpha([\mathbf{0},\mathbf{s}])\Big|^m\le(m!)^pC^m||\mathbf{t}-\mathbf{s}||^{\frac{2p-d(p-1)}{2p}m}.$$

Proof. We fist consider the case $\mathbf{s}\le\mathbf{t}$. Write $[\mathbf{0},\mathbf{t}]\setminus[\mathbf{0},\mathbf{s}]$ into the union of p disjoint boxes. For each $1\le j\le p$, the jth box is of the form $A_1\times\cdots\times A_p$, where $A_l=[0,s_l]$ or $[0,t_l]$ for $l\ne j$, and $A_j=[s_j,t_j]$. By (2.2.7) in Theorem 2.2.3, for $l=1,\cdots,p$

$$\mathbb{E}\Big[\alpha\big(A_l^p\big)^m\Big]=\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\Bigg[\sum_{\sigma\in\Sigma_m}\int_{(A_l)^m_<}du_1\cdots du_m$$
$$\times\prod_{k=1}^m p_{u_k-u_{k-1}}(x_{\sigma(k)}-x_{\sigma(k-1)})\Bigg]^p.$$

For $l \neq j$, we use the estimate given in (2.2.8) and (2.2.9), which gives that

$$\mathbb{E}\Big[\alpha\big(A_l^p\big)^m\Big] \le (m!)^p C^m \qquad m = 1, 2, \cdots \tag{2.2.12}$$

where $C > 0$ represents a constant independent of $\mathbf{s}$, $\mathbf{t}$ and m (To simplify our notation, we allow C to be different from place to place).

For $l = j$, observe that

$$\begin{aligned}
&\int_{(A_j)^m_<} du_1 \cdots du_m \prod_{k=1}^m p_{u_k - u_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)}) \\
&\le \Big(\int_{s_j}^{t_j} p_u(x_{\sigma(1)}) du\Big) \prod_{k=2}^m \int_{[0, t_j - s_j]} p_u(x_{\sigma(k)} - x_{\sigma(k-1)}) du.
\end{aligned}$$

By an estimate modified from the one used for (2.2.8),

$$\begin{aligned}
&\mathbb{E}\Big[\alpha\big(A_j^p\big)^m\Big] \\
&\le (m!)^p \Big\{\int_{\mathbb{R}^d} \Big[\int_{s_j}^{t_j} p_u(x) du\Big]^p dx\Big\}\Big\{\int_{\mathbb{R}^d} dx \Big[\int_0^{t_j - s_j} p_u(x) du\Big]^p dx\Big\}^{m-1}.
\end{aligned}$$

By the estimate (2.2.9)

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Big[\int_0^{t_j - s_j} p_u(x) du\Big]^p dx \\
&\le (2\pi)^{-\frac{d(p-1)}{2}} p^{-d/2} \Big(\frac{2p}{2p - d(p-1)}\Big)^p (t_j - s_j)^{\frac{2p - d(p-1)}{2}}.
\end{aligned}$$

By modifying the estimate used for (2.2.9),

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Big[\int_{s_j}^{t_j} p_u(x) du\Big]^p dx \\
&\le (2\pi)^{-\frac{d(p-1)}{2}} p^{-d/2} \int_{s_j}^{t_j} s^{-\frac{d(p-1)}{2p}} ds \\
&= (2\pi)^{-\frac{d(p-1)}{2}} p^{-d/2} \Big(\frac{2p}{2p - d(p-1)}\Big)^p \Big(t_j^{\frac{2p - d(p-1)}{2p}} - s_j^{\frac{2p - d(p-1)}{2p}}\Big)^p \\
&\le (2\pi)^{-\frac{d(p-1)}{2}} p^{-d/2} \Big(\frac{2p}{2p - d(p-1)}\Big)^p (t_j - s_j)^{\frac{2p - d(p-1)}{2}}.
\end{aligned}$$

Summarizing what we have accomplished,

$$\mathbb{E}\Big[\alpha\big(A_j^p\big)^m\Big] \le (m!)^p C^m (t_j - s_j)^{\frac{2p - d(p-1)}{2} m}. \tag{2.2.13}$$

Thus, by (2.2.11), (2.2.12) and (2.2.13) we have

$$\mathbb{E}\Big[\alpha(A_1\times\cdots\times A_p)^m\Big]\le (m!)^pC^m(t_j-s_j)^{\frac{2p-d(p-1)}{2p}m}.$$

This leads to

$$\mathbb{E}\Big[\alpha\big([\mathbf{0},\mathbf{t}]\big)-\alpha\big([\mathbf{0},\mathbf{s}]\big)\Big]^m\le (m!)^pC^m||\mathbf{t}-\mathbf{s}||^{\frac{2p-d(p-1)}{2p}m}.$$

Observe that when $\mathbf{s}\le\mathbf{t}$,

$$\alpha\big([\mathbf{0},\mathbf{t}]\big)-\alpha\big([\mathbf{0},\mathbf{s}]\big)\ge 0.$$

Hence, we have proved the desired conclusion in the special case when $\mathbf{s}\le\mathbf{t}$.

For the general $\mathbf{s}=(s_1,\cdots,s_p)$ and $\mathbf{t}=(t_1,\cdots,t_p)$, write

$$\mathbf{s}'=(s_1\wedge t_1,\cdots,s_p\wedge t_p),\qquad \mathbf{t}'=(s_1\vee t_1,\cdots,s_p\vee t_p).$$

By monotonicity of the functional $\alpha([\mathbf{0},\mathbf{t}])$ in $\mathbf{t}$,

$$\Big|\alpha\big([\mathbf{0},\mathbf{t}]\big)-\alpha\big([\mathbf{0},\mathbf{s}]\big)\Big|\le\alpha\big([\mathbf{0},\mathbf{t}']\big)-\alpha\big([\mathbf{0},\mathbf{s}']$$

By what we have proved

$$\begin{aligned}\mathbb{E}\Big|\alpha\big([\mathbf{0},\mathbf{t}]\big)-\alpha\big([\mathbf{0},\mathbf{s}]\big)\Big|^m&\le (m!)^pC^m||\mathbf{t}'-\mathbf{s}'||^{\frac{2p-d(p-1)}{2p}m}\\&=(m!)^pC^m||\mathbf{t}-\mathbf{s}||^{\frac{2p-d(p-1)}{2p}m}.\end{aligned}$$

□

Two processes $\{X_t\}_{t\in\Theta}$ and $\{Y_t\}_{t\in\Theta}$ are said to be *modification* to each other, if $\mathbb{P}\{X_t=Y_t\}=1$ for every $t\in\Theta$. By *Kolmogorov's continuity theorem* (Theorem D.7, Appendix) and by Lemma 2.2.4, the process

$$\Big\{\alpha([\mathbf{0},\mathbf{t}]);\quad \mathbf{t}\in[0,T]^p\Big\}$$

has a continuous modification

$$\alpha^T=\Big\{\alpha^T([\mathbf{0},\mathbf{t}])\quad \mathbf{t}\in[0,T]^p\Big\}.$$

For any $T<T'$, we have that $\mathbb{P}\Big\{\alpha^T([\mathbf{0},\mathbf{t}])=\alpha^{T'}([\mathbf{0},\mathbf{t}])\Big\}=1$ for all $\mathbf{t}\in[0,T]^p$. By continuity,

$$\mathbb{P}\Big\{\alpha^T([\mathbf{0},\mathbf{t}])=\alpha^{T'}([\mathbf{0},\mathbf{t}])\ \text{ for all } \mathbf{t}\in[0,T]^p\Big\}=1.$$

Let $0 < T_1 < T_2 < \cdots < T_k < \cdots$ be an increasing sequence such that $T_k \to \infty$ as $k \to \infty$. Then outside a null event, the limit

$$\tilde{\alpha}([\mathbf{0}, \mathbf{t}]) \equiv \lim_{k\to\infty} \alpha^{T_k}([\mathbf{0}, \mathbf{t}])$$

exists for every $\mathbf{t} \in (\mathbb{R}^+)^p$ and, as a function of $\mathbf{t}$, $\tilde{\alpha}([\mathbf{0}, \mathbf{t}])$ is continuous on $(\mathbb{R}^+)^p$. In addition, the process $\Big\{\tilde{\alpha}([\mathbf{0}, \mathbf{t}]);\quad \mathbf{t} \in (\mathbb{R}^+)^p\Big\}$ is a modification of $\Big\{\alpha([\mathbf{0}, \mathbf{t}]);\quad \mathbf{t} \in (\mathbb{R}^+)^p\Big\}$.

By the measure extension theorem, the continuous distribution function $\tilde{\alpha}([\mathbf{0}, \mathbf{t}])$ can be extended uniquely into a (random) measure $\tilde{\alpha}(\cdot)$ on $(\mathbb{R}^+)^p$. We call $\tilde{\alpha}(\cdot)$ the *mutual intersection local time* generated by the independent Brownian motions $W_1(t), \cdots, W_p(t)$.

It is straightforward to see that for any bounded sets $A_1, \cdots, A_p \subset \mathbb{R}^+$,

$$\mathbb{P}\Big\{\alpha(A_1 \times \cdots \times A_p) = \tilde{\alpha}(A_1 \times \cdots \times A_p)\Big\} = 1.$$

To simplify our notation, from now on we use $\alpha(A)$, rather than $\tilde{\alpha}(A)$, to denote this continuous version.

We now turn our attention to establishing properties for the intersection local time $\alpha(A)$.

Proposition 2.2.5 *The measure $\alpha(\cdot)$ is supported on the set*

$$\{(t_1, \cdots, t_p) \in (\mathbb{R}^+)^p;\quad W_1(t_1) = \cdots = W_p(t_p)\}$$

Proof. By continuity of Brownian trajectories, the set

$$\Theta = \{(t_1, \cdots, t_p) \in (\mathbb{R}^+)^p;\quad W_1(t_1) = \cdots = W_p(t_p)\}$$

is a close set. Let $A = [s_1, t_1] \times \cdots \times [s_p, t_p] \subset (\mathbb{R}^+)^p$ be a deterministic box with rational coordinates $s_1, \cdots, s_p; t_1, \cdots, t_p$ and write

$$B_A = \Big\{\omega;\quad A \cap \Theta = \emptyset\Big\}$$

On the event B_A, $\lim_{\epsilon\to 0^+} \alpha_\epsilon(A) = 0$, which implies that $\alpha(A) = 0$. Observe that the fact that Θ is a close set leads to

$$\Theta^c = \bigcup \{A;\quad A \cap \Theta = \emptyset\}$$

where the right hand side is a countable union. Hence, we have that $\alpha(\Theta^c) = 0$. $\square$

Proposition 2.2.6 *For any $a, t > 0$,*

$$\alpha\big([0, at]^p\big) \overset{d}{=} a^{\frac{2p-d(p-1)}{2}} \alpha\big([0, t]^p\big)$$

Proof. By definition and integration variable substitution,

$$\begin{aligned}
\alpha_\epsilon\big([0, at]^p\big) &= a^p \int_{\mathbb{R}^d} dx \prod_{j=1}^p \int_0^t p_\epsilon\big(W_j(as) - x\big) ds \\
&\overset{d}{=} a^p \int_{\mathbb{R}^d} dx \prod_{j=1}^p \int_0^t p_\epsilon\big(\sqrt{a} W_j(s) - x\big) ds \\
&= a^{\frac{2p-dp}{2}} \int_{\mathbb{R}^d} dx \prod_{j=1}^p \int_0^t p_{\epsilon/a}\big(W_j(s) - (x/\sqrt{a})\big) ds \\
&= a^{\frac{2p-d(p-1)}{2}} \int_{\mathbb{R}^d} dx \prod_{j=1}^p \int_0^t p_{\epsilon/a}\big(W_j(s) - x)\big) ds \\
&= \alpha_{\epsilon/a}\big([0, t]^p\big)
\end{aligned}$$

where the equality in law follows from the Brownian scaling. By Theorem 2.2.3, letting $\epsilon \to 0^+$ on the both sides proves the conclusion. $\square$

Our construction of the intersection local time $\alpha(A)$ constitutes the notation

$$\alpha(A) = \int_{\mathbb{R}^d} \left[\int_A \prod_{j=1}^p \delta_x\big(W_j(s_j)\big) ds_1 \cdots ds_m \right] dx. \tag{2.2.14}$$

Recall that in the special case $d = 1$, the occupation measures $\mu_t(\cdot)$ $(t \geq 0)$

$$\mu_t(B) = \int_0^t 1_B\big(W(s)\big) ds \qquad B \subset \mathbb{R}$$

of an 1-dimensional Brownian motion $W(t)$ are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$; and that the family of the associated Radon-Nikodym derivatives formally given as

$$L(t, x) = \int_0^t \delta_x\big(W(s)\big) ds \qquad x \in \mathbb{R} \tag{2.2.15}$$

has a modification jointly continuous in both time variable t and space variable x. We call the continuous random function $L(t, x)$ the *local time* of $W(t)$. Let $L_1(t, x), \cdots, L_p(t, x)$ be the local times of the independent 1-dimensional

Brownian motion $W_1(t), \cdots, W_p(t)$, respectively. We then have that for any $t_1, \cdots, t_p > 0$

$$\alpha_\epsilon\big([0,t_1]\times\cdots\times[0,t_p]\big) = \int_{-\infty}^{\infty}\left[\prod_{j=1}^{p}\int_{-\infty}^{\infty}p_\epsilon(y-x)L_j(t_j,y)dy\right]dx.$$

A deterministic argument by Lemma 2.2.2 establishes the fact that when $d=1$,

$$\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big) = \int_{-\infty}^{\infty}\left[\prod_{j=1}^{p}L_j(t_j,x)\right]dx. \tag{2.2.16}$$

Write

$$h(x_1,\cdots,x_{p-1}) = \int_{\mathbb{R}^d}p_1(-x)\prod_{j=1}^{p-1}p_1\Big(\sum_{k=j}^{p-1}x_k - x\Big)dx.$$

It is easy to verify that $h(\cdot)$ is a probability density on $\mathbb{R}^{d(p-1)}$. When $\epsilon\to 0^+$, the function

$$h_\epsilon = \epsilon^{-d(p-1)}h(\epsilon^{-1}x_1,\cdots,\epsilon^{-1}x_{p-1})$$

approaches to the Dirac function

$$\delta_0(x_1,\cdots,x_{p-1}) = \prod_{j=1}^{p-1}\delta_0(x_j)$$

in distribution. It is straightforward to check that for any bounded set $A\subset(\mathbb{R}^+)^p$,

$$\begin{aligned}\alpha_\epsilon(A) = &\int_A ds_1\cdots ds_m \\ &\times h_\epsilon\big(W_1(s_1)-W_2(s_2),\cdots,W_{p-1}(s_{p-1})-W(s_p)\big).\end{aligned} \tag{2.2.17}$$

This relation justifies the notation introduced in (2.1.2).

We have seen the fundamental role played by Le Gall's moment identity in constructing the intersection local time $\alpha(A)$. In Chapter 3, we shall see an even greater role by this powerful tool in establishing the large deviations for $\alpha([0,t]^p)$. In the following we demonstrate that Le Gall's moment identity takes a simpler form if the Brownian motions run up to exponential times. To this end, we first introduce the following analytic lemma.

Lemma 2.2.7 *For any continuous functions $\varphi_1(t),\cdots,\varphi_m(t)$ satisfying*

$$\int_0^\infty e^{-t}|\varphi_k(t)|dt<\infty \qquad k=1,\cdots,m$$

$$\int_0^\infty dte^{-t}\int_{[0,t]^m_<}ds_1\cdots ds_m\prod_{k=1}^m\varphi_k(s_k-s_{k-1})=\prod_{k=1}^m\int_0^\infty e^{-t}\varphi_k(t)dt$$

where we following the convention $s_0=0$.

Proof. By the substitution

$$t_k=s_k-s_{k-1},\quad (k=1,\cdots,m)\quad t_{m+1}=t-s_m$$

we have

$$\begin{aligned}&\int_0^\infty dte^{-t}\int_{[0,t]^m_<}ds_1\cdots ds_m\prod_{k=1}^m\varphi_k(s_k-s_{k-1})\\&=\int_0^\infty\cdots\int_0^\infty dt_1\cdots dt_{m+1}e^{-t_{m+1}}\prod_{k=1}^m e^{-t_k}\varphi(t_k)\\&=\prod_{k=1}^m\int_0^\infty e^{-t}\varphi_k(t)dt.\end{aligned}$$

□

Let $\tau_1,\cdots\tau_p$ be independent exponential times with parameter 1. We also assume the independence between $\{\tau_1,\cdots\tau_p\}$ and the Brownian motions $\{W_1,\cdots,W_p\}$.

Theorem 2.2.8 *For any any integer $m\ge 1$,*

$$\begin{aligned}&\mathbb{E}\Big[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]\\&=\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p.\end{aligned}\tag{2.2.18}$$

where

$$G(x)=\int_0^\infty e^{-t}p_t(x)dx \qquad x\in\mathbb{R}^d$$

and $x_{\sigma(0)}=0$.

Proof. First, it is easy to see from (2.2.3) that the right hand side of the equation is finite. For any $t_1, \cdots, t_p > 0$, by (2.2.7)

$$\begin{aligned}
&\mathbb{E}\Big[\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big] \\
&= \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \prod_{j=1}^p \sum_{\sigma\in\Sigma_m} \int_{([0,t_j]^m_<} ds_1\cdots ds_m \\
&\qquad\qquad \times \prod_{k=1}^m p_{s_k - s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)})
\end{aligned}$$

By Fubini theorem,

$$\begin{aligned}
&\mathbb{E}\Big[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big] \\
&= \int_0^\infty\cdots\int_0^\infty dt_1\cdots dt_p e^{-(t_1+\cdots+t_p)} \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \\
&\times \prod_{j=1}^p \sum_{\sigma\in\Sigma_m} \int_{([0,t_j]^m_<} ds_1\cdots ds_m \prod_{k=1}^m p_{s_k - s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)}) \\
&= \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \prod_{j=1}^p \sum_{\sigma\in\Sigma_m} \int_0^\infty dt e^{-t} \\
&\times \int_{([0,t]^m_<} ds_1\cdots ds_m \prod_{k=1}^m p_{s_k - s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)}).
\end{aligned}$$

Finally, by Lemma 2.2.7,

$$\begin{aligned}
&\int_0^\infty dt e^{-t} \int_{([0,t]^m_<} ds_1\cdots ds_m \prod_{k=1}^m p_{s_k - s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)}) \\
&= \prod_{k=1}^m \int_0^\infty e^{-t} p_t(x_{\sigma(k)} - x_{\sigma(k-1)}) dt = \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)}).
\end{aligned}$$

$\square$

As an application of Le Gall's moment identity, we prove the following

Theorem 2.2.9 *There is a constant $\lambda > 0$ such that*

$$\mathbb{E}\exp\Big\{\lambda\alpha\big([0,1]^p\big)^{\frac{2}{d(p-1)}}\Big\} < \infty.$$

Proof. Notice that $\tau_{\min} = \min\{\tau_1, \cdots, \tau_p\}$ is exponential with parameter p. By the scaling property given in Proposition 2.2.6 and independence between Brownian motions and exponential times,

$$\begin{aligned}
&\mathbb{E}\Big[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big] \qquad (2.2.19)\\
&\ge \mathbb{E}\Big[\alpha\big([0,\tau_{\min}]^p\big)^m\Big] = \mathbb{E}\tau_{\min}^{\frac{2p-d(p-1)}{2}m}\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\\
&= p^{-\frac{2p-d(p-1)}{2}m-1}\Gamma\Big(1+\frac{2p-d(p-1)}{2}m\Big)\mathbb{E}\alpha\big([0,1]^p\big)^m.
\end{aligned}$$

On the other hand, by (2.2.18) and an argument by Jensen inequality

$$\mathbb{E}\Big[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p] \le (m!)^p\Big(\int_{\mathbb{R}^d} G^p(x)dx\Big)^m. \qquad (2.2.20)$$

In view of (2.2.3), an estimate by Stirling formula based on (2.2.19) and (2.2.20) leads to the bound

$$\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big] \le (m!)^{\frac{d(p-1)}{2}}C^m \qquad m = 1, 2, \cdots.$$

The rest follows from a standard application of Taylor expansion. □

The Brownian motions $W_1(t), \cdots, W_p(t)$ are allowed to started at somewhere rather than the origin. When $W_1(0) = y_1, \cdots, W_p(0) = y_p$, we use $\mathbb{E}_{\bar{y}}$ and $\mathbb{P}_{\bar{y}}$ for the expectation and probability, respectively, of the random path of $\big(W_1(t), \cdots, W_p(t)\big)$. Here we adopt the notation $\bar{y} = (y_1, \cdots, y_p)$. In the important special case $y_1 = \cdots = y_p = 0$, we still use "$\mathbb{E}$" and "$\mathbb{P}$" instead. Under the law $\mathbb{P}_{\bar{y}}$, by the same procedure one can construct the intersection local time $\alpha(A)$. We list some similar properties here.

Similar to the identity in law given in Proposition 2.2.6, for any $a, t > 0$,

$$\mathcal{L}_{\bar{y}}\Big\{\alpha\big([0,at]^p\big)\Big\} = \mathcal{L}_{\bar{y}/\sqrt{a}}\Big\{a^{\frac{2p-d(p-1)}{2}}\alpha\big([0,t]^p\big)\Big\}. \qquad (2.2.21)$$

Under $\mathbb{P}_{\bar{y}}$, the Le Gall' moment identity takes form

$$\begin{aligned}
&\mathbb{E}_{\bar{y}}\Big[\alpha\big(A_1\times\cdots\times A_p\big)^m\Big] \qquad (2.2.22)\\
&= \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \prod_{j=1}^p \sum_{\sigma_j\in\Sigma_m}\int_{(A_j)^m_<} ds_1\cdots ds_m\\
&\qquad\times\prod_{k=1}^m p_{s_k-s_{k-1}}\big(x_{\sigma_j(k)} - x_{\sigma_j(k-1)}\big)
\end{aligned}$$

where $A_1, \cdots, A_p \subset (\mathbb{R}^+)^p$ are bounded, $s_0 = 0$ and $\sigma_j(0) = y_j$ for $j = 1, \cdots, p$.

Taking $A_1 = \cdots = A_p = A$ and applying Hölder inequality to (2.2.22),

$$\mathbb{E}_{\bar{y}}\Big[\alpha\big(A^p\big)^m\Big] \le \prod_{j=1}^p \bigg\{ \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \bigg[\sum_{\sigma_j \in \Sigma_m} \int_{A_<^m} ds_1 \cdots ds_m \\ \times \prod_{k=1}^m p_{s_k - s_{k-1}}\big(x_{\sigma_j(k)} - x_{\sigma_j(k-1)}\big)\bigg]^p \bigg\}^{1/p}.$$

Notice that for each $1 \le j \le p$, the jth factor on the right hand side does not depends on the value of $x_{\sigma_j(0)}$. Consequently, the right hand side is equal to $\mathbb{E}\Big[\alpha\big(A^p\big)^m\Big]$. Thus,

$$\mathbb{E}_{\bar{y}}\Big[\alpha\big(A^p\big)^m\Big] \le \mathbb{E}\Big[\alpha\big(A^p\big)^m\Big] \qquad \forall \bar{y} \in (\mathbb{R}^d)^p. \tag{2.2.23}$$

This inequality shows that for the Brownian motions $W_1(t), \cdots, W_p(t)$, the best strategy to get maximal intersection within the fixed period is to start at the same point.

2.3 Self-intersection local time

Let $W(t)$ be a d-dimensional Brownian motion. The main goal in this section is to construct the random measure $\beta(A)$ on $(\mathbb{R}^+)^p_<$ formally given by (2.1.1) or, equivalently, by

$$\beta(A) = \int_{\mathbb{R}^d} dx \int_A ds_1 \cdots ds_p \prod_{j=1}^p \delta_x\big(W(s_j)\big) \qquad A \subset (\mathbb{R}^+)^p_<. \tag{2.3.1}$$

Let the numbers $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{p-1} < b_p$ be fixed and write

$$W^{(j)}(t) = W(a_j + t) \qquad 0 \le t \le b_j - a_j \quad j = 1, \cdots, p$$

We assign each of the independent Brownian motions $W_1(t), \cdots, W_p(t)$ an initial distribution μ with the density

$$h(x) = C_d e^{-|x|} \qquad x \in \mathbb{R}^d$$

where $C_d > 0$ normalizes $h(x)$ into a density function.

Write $T = [0, b_1 - a_1] \times \cdots \times [0, b_p - a_p]$. We view the random fields

$$\Big(W^{(1)}(t_1), \cdots, W^{(p)}(t_p)\Big) \quad \text{and} \quad \Big(W_1(t_1), \cdots, W_p(t_p)\Big) \qquad (t_1, \cdots, t_p) \in T$$

as two random variables with values in the space $C\{T,(\mathbb{R}^d)^p\}$ and with the distributions, respectively, $\gamma_1(dw_1,\cdots,dw_p)$ and $\gamma_2(dw_1,\cdots,dw_p)$.

Lemma 2.3.1 *$\gamma_1(\cdot)$ is absolutely continuous with respect to $\gamma_2(\cdot)$ with the associated Random-Nikodym derivative*

$$\Theta(w_1,\cdots,w_p)=C_d^{-p}\exp\Big\{\sum_{j=1}^p|w_j(0)|\Big\}p_{a_1}\big(w_1(0)\big)\times\prod_{j=2}^p p_{a_j-b_{j-1}}\big(w_j(0)-w_{j-1}(b_{j-1}-a_{j-1})\big). \tag{2.3.2}$$

In particular

$$\int_{C\{T,(\mathbb{R}^d)^p\}}\Theta^2(w_1,\cdots,w_p)\gamma_2(dw_1,\cdots,dw_p)<\infty. \tag{2.3.3}$$

Proof. For each $1\le j\le p$, let $0=s_0^{(j)}<s_1^{(j)}<\cdots<s_{n_j}^{(j)}=b_j-a_j$ be a partition of $[0,b_j-a_j]$. It is straightforward to verify that the random vector

$$\Big(W^{(1)}(s_0^{(1)}),\cdots,W^{(1)}(t_{n_1}^{(1)});\cdots;W^{(p)}(s_0^{(p)}),\cdots,W^{(p)}(t_{n_p}^{(p)})\Big)$$

has the probability density

$$\begin{aligned}&f(x_0^{(1)},\cdots,x_{n_1}^{(1)};\cdots;x_0^{(p)},\cdots,x_{n_p}^{(p)})\\&=\prod_{j=1}^p p_{a_j-b_{j-1}}(x_0^{(j)}-x_{n_{j-1}}^{(j-1)})\prod_{k=1}^{n_j}p_{s_k^{(j)}-s_{k-1}^{(j)}}(x_k^{(j)}-x_{k-1}^{(j)})\end{aligned}$$

where $b_0=0$ and $x_{n_0}^{(0)}=0$; and that the random vector (with the given initial distribution)

$$\Big(W_1(s_0^{(1)}),\cdots,W_1(t_{n_1}^{(1)});\cdots;W_p(s_0^{(p)}),\cdots,W_p(t_{n_p}^{(p)})\Big)$$

has the probability density

$$\begin{aligned}&g(x_0^{(1)},\cdots,x_{n_1}^{(1)};\cdots;x_0^{(p)},\cdots,x_{n_p}^{(p)})\\&=\prod_{j=1}^p C_d e^{-|x_0^{(j)}|}\prod_{k=1}^{n_j}p_{s_k^{(j)}-s_{k-1}^{(j)}}(x_k^{(j)}-x_{k-1}^{(j)}).\end{aligned}$$

Hence,

$$\frac{f(x_0^{(1)},\cdots,x_{n_1}^{(1)};\cdots;x_0^{(p)},\cdots,x_{n_p}^{(p)})}{g(x_0^{(1)},\cdots,x_{n_1}^{(1)};\cdots;x_0^{(p)},\cdots,x_{n_p}^{(p)})}$$
$$= C_d^{-p}\exp\Big\{\sum_{j=1}^p |x_0^{(j)}|\Big\} p_{a_1}(x_0^{(1)})\prod_{j=2}^p p_{a_j-b_{j-1}}(x_0^{(j)}-x_{n_{j-1}}^{(j-1)}).$$

This leads to the identity

$$\gamma_1(A)=\int_A \Theta(w_1,\cdots,w_p)\gamma_2(dw_1,\cdots,dw_p)$$

for any $A\subset C\big\{T,(\mathbb{R}^d)^p\big\}$ of the form

$$A=\Big\{(w_1,\cdots,w_p);\ \big(w_1(s_0^{(1)}),\cdots,w_1(t_{n_1}^{(1)});\cdots;w_p(s_0^{(p)}),\cdots,w_p(t_{n_p}^{(p)})\big)\in B\Big\}.$$

Notice that $\Theta(w_1,\cdots,w_p)$ does not depend on the time set we choice. By standard measure extension, the identity holds for all $A\subset C\big\{T,(\mathbb{R}^d)^p\big\}$.

Finally, a straightforward calculation gives that

$$\int_{C\{T,(\mathbb{R}^d)^p\}}\Theta^2(w_1,\cdots,w_p)\gamma_2(dw_1,\cdots,dw_p)=\mathbb{E}\Theta^2(W_1,\cdots,W_p)<\infty.$$

□

For each $\epsilon>0$, define the random measure $\beta_\epsilon(\cdot)$ on $(\mathbb{R}^+)^p_<$:

$$\beta_\epsilon(A)=\int_{\mathbb{R}^d}\bigg[\int_A ds_1\cdots ds_p\prod_{j=1}^p p_\epsilon\big(W(s)-x\big)\bigg]dx \qquad A\subset(\mathbb{R}^+)^p_<.$$

A measure on a complete metric space is called *Radon measure* if the measure value of every compact set is finite.

Theorem 2.3.2 *Assume $p(d-2)<d$. With probability 1 there exists a Radon measure $\beta(\cdot)$ on $(\mathbb{R}^+)^p_<$ such that,*

(1) For any compact subset of $(\mathbb{R}^+)^p_<$ of the form $A_1\times\cdots\times A_p$, where $A_1,\cdots,A_p$ are finite close intervals,

$$\lim_{\epsilon\to 0^+}\mathbb{E}\big|\beta_\epsilon(A_1\times\cdots\times A_p)-\beta(A_1\times\cdots\times A_p)\big|^m=0 \tag{2.3.4}$$

for all $m>0$.

(2). The measure is supported on:

$$\big\{(s_1,\cdots,s_p)\in(\mathbb{R}^+)^p_<;\quad W(s_1)=\cdots=W(s_p)\big\}.$$

In addition, $\beta\big(\{s_j=t\}\big)=0$ *for any* $1\le j\le p$ *and any* $t\ge 0$.

Proof. Let the numbers $0\le a_1<b_1<\cdots<a_p<b_p$ be given and write

$$I=[\mathbf{a},\mathbf{b}]=[a_1,b_1]\times\cdots\times[a_p,b_p]. \tag{2.3.5}$$

We first construct a random measure $\beta^I(\cdot)$ on I which will appear as the limitation of $\beta(\cdot)$ on I. Notice that the fact that for any $t>0$,

$$\Big\{\beta_\epsilon(t+A);\quad A\subset(\mathbb{R}^+)^p_<\Big\}\overset{d}{=}\Big\{\beta_\epsilon(A)\quad A\subset(\mathbb{R}^+)^p_<\Big\}$$

where $t+A=\big\{(t+s_1,\cdots,t+s_p);\quad(s_1,\cdots,s_p)\in A\big\}$. We may assume $a_1>0$, for otherwise we may consider $[1+a_1,1+b_1]\times\cdots\times[1+a_p,1+b_p]$ instead.

We adopt the notations used in Lemma 2.3.1. Let $\mathbf{t}\in I$ and define the function ξ_ϵ on $C\big\{I,(\mathbb{R}^d)^p\big\}$ as

$$\xi_\epsilon(w_1,\cdots,w_p)=\int_{\mathbb{R}^d}\bigg[\prod_{j=1}^p\int_0^{t_j-a_j}p_\epsilon\big(w_j(s)-x\big)ds\bigg]dx.$$

By time shifting one can see that

$$\beta_\epsilon([\mathbf{a},\mathbf{t}])=\xi_\epsilon\big(W^{(1)},\cdots,W^{(p)}\big).$$

By Lemma 2.3.1, for any $\epsilon,\epsilon'>0$,

$$\begin{aligned}
&\mathbb{E}\Big|\beta_\epsilon\big([\mathbf{a},\mathbf{t}]\big)-\beta_{\epsilon'}\big([\mathbf{a},\mathbf{t}]\big)\Big|^m\\
&=\int_{C\{T,(\mathbb{R}^d)^p\}}\big|\xi_\epsilon(w_1,\cdots,w_p)-\xi_{\epsilon'}(w_1,\cdots,w_p)\big|^m\\
&\qquad\qquad\times\Theta(w_1,\cdots,w_p)\gamma_2(dw_1,\cdots,dw_p)\\
&\le\bigg\{\int_{C\{T,(\mathbb{R}^d)^p\}}\Theta^2(w_1,\cdots,w_p)\gamma_2(dw_1,\cdots,dw_p)\bigg\}^{1/2}\\
&\times\Big\{\mathbb{E}\big|\alpha_\epsilon([\mathbf{0},\mathbf{t}-\mathbf{a}])-\alpha_{\epsilon'}([\mathbf{0},\mathbf{t}-\mathbf{a}])\big|^{2m}\Big\}^{1/2}.
\end{aligned}$$

By Theorem 2.2.3, therefore, there is a non-negative random variable $\beta^I\big([\mathbf{a},\mathbf{t}]\big)$, such that

$$\lim_{\epsilon\to 0^+}\beta_\epsilon\big([\mathbf{a},\mathbf{t}]\big)=\beta^I\big([\mathbf{a},\mathbf{t}]\big) \tag{2.3.6}$$

in all positive moments.

For any $\mathbf{s}, \mathbf{t} \in [\mathbf{a}, \mathbf{b}]$, by Lemma 2.3.1 and a procedure same as above,

$$\begin{aligned}
&\mathbb{E}\Big|\beta_\epsilon\big([\mathbf{a},\mathbf{t}]\big) - \beta_\epsilon\big([\mathbf{a},\mathbf{s}]\big)\Big|^m \\
&\le \bigg\{\int_{C\left\{T,(\mathbb{R}^d)^p\right\}} \Theta^2(w_1,\cdots,w_p)\gamma_2(dw_1,\cdots,dw_p)\bigg\}^{1/2} \\
&\times \Big\{\mathbb{E}\Big|L_\epsilon(\mathbf{t}-\mathbf{a},0) - L_\epsilon(\mathbf{s}-\mathbf{a},0)\Big|^{2m}\Big\}^{1/2}.
\end{aligned}$$

By Lemma 2.2.4, for any $M > 0$ there is a $C_I > 0$ such that for any $\mathbf{s}, \mathbf{t} \in [0, M]^p$

$$\mathbb{E}\big|\beta^I\big([\mathbf{a},\mathbf{s}]\big) - \beta^I\big([\mathbf{a},\mathbf{t}]\big)\big|^m \le (m!)^p C_I^m ||\mathbf{s}-\mathbf{t}||^{\frac{2p-d(p-1)}{2}m} \qquad m = 1, 2, \cdots.$$

By Kolmogorov continuity theorem (Theorem D.7, Appendix), the process $\Big\{\beta^I\big([\mathbf{a},\mathbf{t}]\big); \;\; \mathbf{t} \in I\Big\}$ has a continuous modification (which still denoted by $\beta^I\big([\mathbf{a},\mathbf{t}]\big)$). The continuous distribution function $\beta^I\big([\mathbf{a},\mathbf{t}]\big)$ then generates a Borel measure $\beta^I(A)$ on I, which is extended to a measure on $(\mathbb{R}^+)^p$ but supported by I. Clearly, the continuity of $\beta^I\big([\mathbf{a},\mathbf{t}]\big)$ leads to $\beta^I(\{s_j = t\}) = 0$ for any $t \ge 0$ and $1 \le j \le p$. In addition, an argument used in the proof of Proposition 2.2.5 shows that the measure $\beta^I(\cdot)$ is supported on

$$\big\{(s_1, \cdots, s_p) \in (\mathbb{R}^+)^p_<; \;\; W(s_1) = \cdots = W(s_p)\big\}.$$

Take a countable collections $\{I_k\}_{k\ge 1}$ of compact rectangles of the form given in (2.3.5) such that

(i) $(\mathbb{R}^+)^p_< = \displaystyle\bigcup_{k=1}^\infty I_k$;

(ii) if $j \ne k$, $I_j \cap I_k$ is contained in a finite union of "hyperplanes" $\{s_j = t\}$;

(iii) any compact set of $(\mathbb{R}^+)^p$ intersects only a finite number of the rectangles I_k.

By our discussion, for each k, there is a finite measure $\beta^{I_k}(\cdot)$ supported on I_k and on the time set on p-multiple points such that $\beta^{I_k}(\cdot)$ does not charge the hyperplane, and that as $\epsilon \to 0^+$, $\beta_\epsilon(I') \longrightarrow \beta^{I_k}(I')$ in all moments for any sub-rectangle $I' \subset I_k$. Define the measure β on $(\mathbb{R}^+)^p_<$ by

$$\beta(A) = \sum_{k=1}^\infty \beta^{I_k}(A) \qquad A \subset (\mathbb{R}^+)^p_<.$$

One can easily see how (2.3.4) follows from (i) and (ii). Finally, the property (iii) ensures that $\beta(\cdot)$ is a Radon measure. □

The random measure $\beta(A)$ is called *p-multiple self-intersection local time* of the Brownian motion $W(t)$.

Similar to (2.2.17)

$$\beta_\epsilon(A) = \int_A ds_1 \cdots ds_p \times h_\epsilon\big(W(s_1) - W_2(s_2), \cdots, W(s_{p-1}) - W(s_p)\big) \tag{2.3.7}$$

which justifies the notation given in (2.1.1).

From the proof of Theorem 2.3.2, we have seen some connections between the mutual intersection local time $\alpha(\cdot)$ and the self-intersection local time $\beta(\cdot)$. Compared with Proposition 2.2.6 the following proposition shows that $\beta(\cdot)$ has a scaling rate same as $\alpha(\cdot)$.

Proposition 2.3.3 *Assume $p(d-2) < d$. For any $t > 0$, and $A \in (\mathbb{R}^+)^p_<$,*

$$\beta(tA) \stackrel{d}{=} t^{\frac{2p-d(p-1)}{2}} \beta(A) \tag{2.3.8}$$

$$\beta(t+A) \stackrel{d}{=} \beta(A) \tag{2.3.9}$$

where $t + A = \{(t+s_1, \cdots, t+s_p); \quad (s_1, \cdots, s_p) \in A\}$.

Proof. By a standard argument of monotonic measure extension, we need only to consider the case when A is of the form given in (2.3.5). Indeed, by variable substitution,

$$\begin{aligned}
\beta_\epsilon(tA) &= t^p \int_{\mathbb{R}^d} dx \prod_{j=1}^p \int_{a_j}^{b_j} p_\epsilon\big(W(ts) - x\big) ds \\
&\stackrel{d}{=} t^p \int_{\mathbb{R}^d} dx \prod_{j=1}^p \int_{a_j}^{b_j} p_\epsilon\big(\sqrt{t}W(s) - x\big) ds \\
&= t^{p-\frac{dp}{2}} \int_{\mathbb{R}^d} dx \prod_{j=1}^p \int_{a_j}^{b_j} p_{\epsilon/\sqrt{t}}\big(W(s) - (x/\sqrt{t})\big) ds \\
&= t^{\frac{2p-d(p-1)}{2}} \int_{\mathbb{R}^d} dx \prod_{j=1}^p \int_{a_j}^{b_j} p_{\epsilon/\sqrt{t}}\big(W(s) - x)\big) ds \\
&= t^{\frac{2p-d(p-1)}{2}} \beta_{\epsilon/\sqrt{t}}(A)
\end{aligned}$$

where the equality in law is implied by the Brownian scaling. Thus, (2.3.8) follows from part (1) of Theorem 2.3.2. The proof of (2.3.9) is similar, where the key ingredient is the increment stationary of Brownian motion and translation invariance of the Lebesgue integral on $\mathbb{R}^d$. $\square$

The next proposition shows that the connection is even more direct when it comes the double intersections.

Proposition 2.3.4 *Assume that $p = 2$ and $d \le 3$. For any $0 \le a < b < c$,*

$$\beta\big([a,b]\times[b,c]\big) \overset{d}{=} \alpha([0,b-a]\times[0,c-b]\big).$$

Proof. Let $b < b' < c$ and notice that as $p = 2$,

$$\begin{aligned}
\beta_\epsilon\big([a,b]\times[b',c]\big) &= \int_a^b \int_b^{\prime c} h_\epsilon\Big(\big(W(r)-W(b)\big)-\big(W(s)-W(b)\big)\Big)dsdr \\
&= \int_{b'-b}^{b'-a}\int_0^{c-b'} h_\epsilon\Big(\big(W(b'-r)-W(b')\big)-\big(W(b'+s)-W(b')\big)\Big)dsdr \\
&= \int_{b'-b}^{b'-a}\int_0^{c-b'} h_\epsilon\big(W_1(r)-W_2(s)\big)dsdr
\end{aligned}$$

where $W_1(r) = -\Big(w(b')-W(b'-r)\Big)$ $(0 \le r \le b'-a)$ and $W_2(s) = W(b'+s)-W(b)$ $(0 \le s \le c-b')$ are independent Brownian motions. Letting $\epsilon \to 0^+$ gives

$$\beta\big([a,b]\times[b',c]\big) \overset{d}{=} \alpha([b'-b,b'-a]\times[0,c-b']\big).$$

Letting $b' \to b^+$ leads to the desired conclusion. $\square$

In another special case $d = 1$, the self-inter section local time can be written in terms of the interal of the local time $L(t,x)$ of the 1-dimensional Brownian motion $W(t)$. For example, by an analysis (omitted here) similar to the one used for (2.2.16) we have

Proposition 2.3.5 *As $d = 1$ and $p \ge 2$,*

$$\beta([0,t]^p_<\big) = \frac{1}{p!}\int_{-\infty}^{\infty} L^p(t,x)dx \qquad t > 0.$$

Here comes a bad news. Sharply contrary to the 1-dimensional case, the self-intersection local time of a multi-dimensional Brownian motion can blow up even in a finite time period.

Proposition 2.3.6 *As $d \ge 2$, for any $0 \le a < b$*

$$\beta\big([a,b]^p_<\big) = \infty \qquad a.s.$$

Proof. By Proposition 2.3.3, we may only consider the case $a = 0$ and $b = 1$. Set

$$I = I^0_0 = \Big[0, \frac{1}{2p}\Big] \times \Big[\frac{2}{2p}, \frac{3}{2p}\Big] \times \cdots \times \Big[\frac{2(p-1)}{2p}, \frac{2p-1}{2p}\Big]$$

and more generally for any $k \ge 0$, $l = 0, 1, \cdots, 2^k - 1$

$$I^k_l = l2^{-k} + 2^{-k} I^0_0 = \Big[l2^{-k} + \frac{1}{2p}2^{-k}\Big] \times \cdots \times \Big[l2^{-k}\frac{2(p-1)}{2p}, \frac{2p-1}{2p}2^{-k}\Big].$$

Then for each k, the random variables $\beta(I^k_l)$ $(l = l = 0, 1, \cdots, 2^k - 1)$ are independent and identically distributed. By Proposition 2.3.3,

$$\beta(I^k_l) \overset{d}{=} 2^{-k\frac{2p-d(p-1)}{2}} \beta(I).$$

It follows that

$$\mathbb{E}\Big[\sum_{l=0}^{2^k-1} \beta(I^k_l)\Big] = 2^{\frac{(d-2)(p-1)}{2}k} \mathbb{E}\big(\beta(I)\big)$$

$$\mathrm{Var}\Big[\sum_{l=0}^{2^k-1} \beta(I^k_l)\Big] = 2^{-(2p-d(p-1)-1)k} \mathrm{Var}\big(\beta(I)\big).$$

By the fact that $2p - d(p-1) - 1 > 0$,

$$\sum_{k=0}^{\infty} \Big| \sum_{l=0}^{2^k-1} \beta(I^k_l) - \mathbb{E}\Big[\sum_{l=0}^{2^k-1} \beta(I^k_l)\Big]\Big| \le \sum_{k=1}^{\infty} \Big\{\mathrm{Var}\Big[\sum_{l=0}^{2^k-1} \beta(I^k_l)\Big]\Big\}^{1/2} < \infty.$$

Consequently,

$$\sum_{k=0}^{\infty} \sum_{l=0}^{2^k-1} \beta(I^k_l) = \infty \qquad a.s.$$

as $d \ge 2$. Finally, the conclusion follows from the fact that $I^k_l \subset [0,1]^p_<$ and and that the intersection of any two members of the family $\{I^k_l\}$ is contained in a finite union of "hyperplanes" $\{s_j = t\}$. □

The reason behind this problem is that when $d \ge 2$, the short range intersection is too strong compared with long range intersection. In the case $d = 2$, a way to fix it is the renormalization method which will be discussed in the next section.

2.4 Renormalization

In this section, we consider the case $d = p = 2$. Because of the problem pointed out in Proposition 2.3.6, the polymer models suggested in (2.1.3) and (2.1.4) are not mathematically defined. In this section we construct so called *renormalized self-intersection local time* $\gamma\big([0,t]_<\big)$ formally written as

$$\gamma\big([0,t]^2_<\big) = \iint_{\{0\le r<s\le t\}} \delta_0\big(W(r)-W(s)\big)drds \qquad (2.4.1)$$
$$- \mathbb{E}\iint_{\{0\le r<s\le t\}} \delta_0\big(W(r)-W(s)\big)drds.$$

The idea is that if we replace $\beta\big([0,t]^2_<\big)$ by $\gamma\big([0,t]_<\big)$ in (2.1.3) and (2.1.4) (of course, the normalizing constants $\widehat{C}$ and $\widetilde{C}$ have to be redefined properly), the models keep intact in physics.

Let $t > 0$ be fixed and set

$$A_l^k = \Big[\frac{2l}{2^{k+1}}t, \frac{2l+1}{2^{k+1}}t\Big) \times \Big[\frac{2l+1}{2^{k+1}}t, \frac{2l+2}{2^{k+1}}t\Big) \qquad (2.4.2)$$
$$l = 0,1,\cdots,2^k-1 \quad k = 0,1,\cdots$$

(See the Figure 2.1).

Theorem 2.4.1 *The series*

$$\sum_{k=0}^\infty \bigg\{\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k) - \mathbb{E}\beta(A_l^k)\Big)\bigg\}$$

convergences with probability 1 and in $\mathcal{L}^2(\Omega,\mathcal{A},\mathbb{P})$.

Proof. By Lemma 2.3.3 and Lemma 2.3.4 we have that

$$\beta(A_l^k) \overset{d}{=} 2^{-(k+1)}\beta\big([0,t]\times[t,2t]\big) \overset{d}{=} 2^{-(k+1)}\alpha\big([0,t]^2\big). \qquad (2.4.3)$$

In addition, for each $k \ge 0$, the finite sequence

$$\beta(A_l^k), \qquad l = 0,1,\cdots,2^k-1$$

is independent. Consequently,

$$\mathrm{Var}\bigg(\sum_{l=0}^{2^k-1}\beta(A_l^k)\bigg) = 2^{-k-2}\mathrm{Var}\Big(\alpha\big([0,t]^2\big)\Big) = C2^{-k}.$$

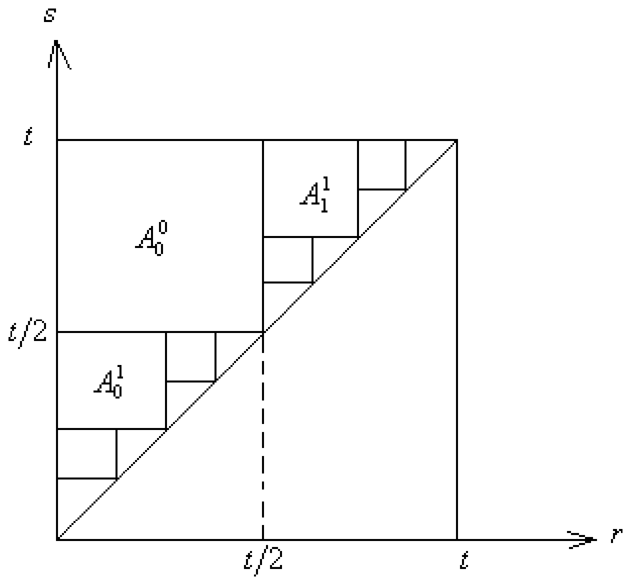

Figure 2.1: triangular approximation

Finally, the conclusion follows from the estimate

$$\begin{aligned}&\bigg\{\mathbb{E}\bigg[\sum_{k=0}^{\infty}\bigg|\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k)-\mathbb{E}\beta(A_l^k)\Big)\bigg|\bigg]^2\bigg\}^{1/2}\\ &\le\sum_{k=0}^{\infty}\bigg\{\mathrm{Var}\bigg(\sum_{l=0}^{2^k-1}\beta(A_l^k)\bigg)\bigg\}^{1/2}<\infty.\end{aligned}$$

□

The approach used in the proof is called *triangular approximation* , due to the geometric intuition from Figure 2.1.

Write

$$\gamma\big([0,t]_<^2\big)=\sum_{k=0}^{\infty}\bigg\{\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k)-\mathbb{E}\beta(A_l^k)\Big)\bigg\}\qquad t\ge 0. \tag{2.4.4}$$

By Lemma 2.3.3, one can see that $\mathbb{E}\gamma\big([0,t]_<^2\big)=0$ and that for any $a>0$,

$$\gamma\big([0,at]_<^2\big)\overset{d}{=}a\gamma\big([0,t]_<^2\big). \tag{2.4.5}$$

We shall use the notation $\gamma(A) = \beta(A) - \mathbb{E}\beta(A)$ and some related properties without further notification whenever it makes obvious sense. For example, if $t' < t$, then by performing the same approximation on the triangle $[t', t]^2_<$ we can define $\gamma\big([t', t]^2_<\big)$. By Proposition 2.3.3, one can verify that

$$\gamma\big([t', t]^2_<\big) \stackrel{d}{=} \gamma\big([0, t - t']^2_<\big). \tag{2.4.6}$$

In the rest of the section, we solve the following two problems: First, we establish the exponential integrability for $\gamma\big([0,t]^2_<\big)$, in order to give mathematical justification to the polymer models in (2.1.3) and (2.1.4) (with $\beta\big([0,t]^2_<\big)$ being replaced by $\gamma\big([0,t]^2_<\big)$). Second, we prove that the family

$$\Big\{\gamma\big([0,t]^2_<\big); \quad t \ge 0\Big\}$$

yields a continuous version.

With regard to our first task, we may take $t = 1$ in following theorem, due to the scaling property given in (2.4.5).

Theorem 2.4.2 *Let $\gamma([0,t]^2_<)$ be defined by (2.4.4).*

(1). For some $\lambda > 0$

$$\mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]^2_<\big)\Big\} < \infty. \tag{2.4.7}$$

(2) For every $\lambda > 0$

$$\mathbb{E}\exp\Big\{-\lambda\gamma\big([0,1]^2_<\big)\Big\} < \infty. \tag{2.4.8}$$

Proof. We first prove (2.4.7). Again, our approach is the triangular approximation based on Figure 2.1 with $t = 1$.

Let A^k_l be defined by (2.4.2) with $t = 1$. By (2.4.3) one can see that

$$\gamma(A^k_l) \stackrel{d}{=} 2^{-(k+1)}\Big\{\alpha\big([0,1]^2 - \mathbb{E}\alpha\big([0,1]^2\Big\}.$$

By Theorem 2.2.9 (with $d = p = 2$) there is a $\lambda_1 > 0$ such that

$$\mathbb{E}\exp\Big\{\lambda_1\alpha\big([0,1]^2\big)\Big\} < \infty.$$

By Taylor expansion, therefore, there is a $C > 0$ such that for sufficiently small $\lambda > 0$

$$\mathbb{E}\exp\Big\{\lambda\Big(\alpha\big([0,1]^2\big) - \mathbb{E}\alpha\big([0,1]^2\big)\Big)\Big\} \le e^{C\lambda^2}.$$

Fix $a \in (0,1)$ and a small $\lambda > 0$. For each $N \geq 1$ set

$$b_1 = 2\lambda, \quad b_N = 2\lambda \prod_{j=2}^{N} \Big(1 - 2^{-a(j-1)}\Big) \qquad N = 2, 3, \cdots .$$

By Hölder inequality

$$\begin{aligned}
&\mathbb{E} \exp\bigg\{ b_N \sum_{k=0}^{N} \sum_{l=0}^{2^k-1} \Big(\beta(A_l^k) - \mathbb{E}\beta(A_l^k)\Big)\bigg\} \\
&\leq \bigg[\mathbb{E} \exp\bigg\{ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} \Big(\beta(A_l^k) - \mathbb{E}\beta(A_l^k)\Big)\bigg\}\bigg]^{1-2^{-a(N-1)}} \\
&\times \bigg[\mathbb{E} \exp\bigg\{ 2^{a(N-1)} b_N \sum_{l=0}^{2^N-1} \Big(\beta(A_l^N) - \mathbb{E}\beta(A_l^N)\Big)\bigg\}\bigg]^{2^{-a(N-1)}} \\
&\leq \mathbb{E} \exp\bigg\{ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} \Big(\beta(A_l^k) - \mathbb{E}\beta(A_l^k)\Big)\bigg\} \\
&\times \bigg[\mathbb{E} \exp\bigg\{ 2^{a(N-1)} b_N \Big(\beta(A_0^N) - \mathbb{E}\beta(A_0^N)\Big)\bigg\}\bigg]^{2^{(1-a)(N-1)}} .
\end{aligned}$$

Notice that $b_N \leq 2\lambda$. We have

$$\begin{aligned}
&\mathbb{E} \exp\Big\{ 2^{a(N-1)} b_N \Big(\beta(A_0^N) - \mathbb{E}\beta(A_0^N)\Big)\Big\} \\
&= \mathbb{E} \exp\Big\{ 2^{-N-1+a(N-1)} b_N \Big(\alpha\big([0,1]^2\big) - \mathbb{E}\alpha\big([0,1]^2\big)\Big)\Big\} \\
&\leq \exp\Big\{ C b_N^2 2^{-2N+2a(N-1)}\Big\}.
\end{aligned}$$

Summarizing what we have

$$\begin{aligned}
&\mathbb{E} \exp\bigg\{ b_N \sum_{k=0}^{N} \sum_{l=0}^{2^k-1} \Big(\beta(A_l^k) - \mathbb{E}\beta(A_l^k)\Big)\bigg\} \\
&\leq \mathbb{E} \exp\bigg\{ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} \Big(\beta(A_l^k) - \mathbb{E}\beta(A_l^k)\Big)\bigg\} \exp\Big\{ C' 2^{(a-1)N}\Big\}.
\end{aligned}$$

Repeating above procedure gives

$$\begin{aligned}
&\mathbb{E}\exp\Big\{b_N\sum_{k=0}^{N}\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k)-\mathbb{E}\beta(A_l^k)\Big)\Big\}\\
&\le\exp\Big\{C'\sum_{k=0}^{N}2^{(a-1)k}\Big\}\le\exp\Big\{C'\Big(1-2^{a-1}\Big)^{-1}\Big\}<\infty.
\end{aligned}$$

Observe that

$$b_\infty=2\lambda\prod_{j=2}^{\infty}\Big(1-2^{-a(j-1)}\Big)>0.$$

By Fatou lemma, letting $N\to\infty$ we have

$$\mathbb{E}\exp\Big\{b_\infty\gamma\big([0,1]_<^2\big)\Big\}\le\exp\Big\{C'\Big(1-2^{a-1}\Big)^{-1}\Big\}<\infty.$$

Similar to the the above argument for (2.4.7), one can prove that there is a $\bar\lambda$ such that

$$\mathbb{E}\exp\Big\{-\lambda\gamma\big([0,1]_<^2\big)\Big\}<\infty\qquad\forall 0\le\lambda\le\bar\lambda. \tag{2.4.9}$$

Define

$$Z_t=-\frac{1}{2\pi}t\log t-\gamma\big([0,t]_<^2\big)\qquad t\ge 0. \tag{2.4.10}$$

For any $s,t>0$,

$$\begin{aligned}
Z_{s+t}&=-\frac{1}{2\pi}(s+t)\log(s+t)-\gamma\big([0,s]_<^2\big)-\gamma\big([s,s+t]_<^2\big)\\
&\quad-\beta\big([0,s]\times[s,s+t]\big)+\mathbb{E}\beta\big([0,s]\times[s,s+t]\big)\\
&\le-\frac{1}{2\pi}(s+t)\log(s+t)-\gamma\big([0,s]_<^2\big)\\
&\quad-\gamma\big([s,s+t]_<^2\big)+\mathbb{E}\beta\big([0,s]\times[s,s+t]\big).
\end{aligned}$$

By Proposition 2.3.4

$$\begin{aligned}
&\mathbb{E}\beta\big([0,s]\times[s,s+t]\big)=\mathbb{E}\alpha\big([0,s]\times[0,t]\big)\\
&=\int_{\mathbb{R}^2}dx\Big[\int_0^s p_u(x)du\Big]\Big[\int_0^t p_u(x)du\Big]=\int_0^s\int_0^t p_{u+v}(0)dudv
\end{aligned}$$

where the second equality follows from the Le Gall's moment formula in the special case $d=p=2$ and $m=1$. By a straightforward calculation,

$$\begin{aligned}
&\int_0^s\int_0^t p_{u+v}(0)dudv=\frac{1}{2\pi}\int_0^s\int_0^t\frac{1}{u+v}dudv\\
&=\frac{1}{2\pi}\Big[(s+t)\log(s+t)-s\log s-t\log t\Big].
\end{aligned}$$

Summarizing our argument,

$$Z_{s+t} \le Z_s + Z'_t \tag{2.4.11}$$

where

$$Z'_t = -\frac{1}{2\pi} t \log t - \gamma\big([s, s+t]^2_<\big)$$

is independent of Z_s and $Z'_t \stackrel{d}{=} Z_t$. This means that Z_t is sub-additive. Consequently, (2.4.8) follows from Theorem 1.3.5. □

To our second task, we first prove

Lemma 2.4.3 *For any $T > 0$ there is a $c = c(T) > 0$ such that*

$$\sup_{\substack{s,t\in[0,T]\\ s\neq t}} \mathbb{E}\exp\left\{c\frac{\big|\gamma\big([0,t]^2_<\big) - \gamma\big([0,s]^2_<\big)\big|}{|t-s|^{1/2}}\right\} < \infty.$$

Proof. For $0 \le s < t \le T$,

$$\gamma\big([0,t]^2_<\big) - \gamma\big([0,s]^2_<\big) = \gamma\big([s,t]^2_<\big) + \gamma\big([0,s]\times[s,t]\big).$$

Notice that

$$\gamma\big([s,t]^2_<\big) \stackrel{d}{=} (t-s)\gamma\big([0,1]^2_<\big).$$

By Theorem 2.4.2, theta is a $c_1 > 0$ such that

$$\sup_{\substack{s,t\in[0,T]\\ s<t}} \mathbb{E}\exp\left\{c_1\frac{\big|\gamma\big([s,t]^2_<\big)\big|}{|t-s|^{1/2}}\right\} < \infty.$$

In addition, by Proposition 2.3.4

$$\gamma\big([0,s]\times[s,t]\big) \stackrel{d}{=} \alpha\big([0,s]\times[0,t-s]\big) - \mathbb{E}\alpha\big([0,s]\times[0,t-s]\big).$$

By (2.2.11),

$$\begin{aligned}&\mathbb{E}\Big[\alpha\big([0,s]\times[0,t-s]\big)^m\Big] \le \Big\{\mathbb{E}\Big[\alpha\big([0,s]^2\big)^m\Big]\Big\}^{1/2}\Big\{\mathbb{E}\Big[\alpha\big([0,t-s]^2\big)^m\Big]\Big\}^{1/2}\\ &= s^{m/2}(t-s)^{m/2}\mathbb{E}\Big[\alpha\big([0,1]^2\big)^m\Big] \le T^{m/2}(t-s)^{m/2}\mathbb{E}\Big[\alpha\big([0,1]^2\big)^m\Big]\end{aligned}$$

for $m = 1, 2, \cdots$. By Theorem 2.2.9, there is a $c_2 > 0$ such that

$$\sup_{\substack{s,t\in[0,T]\\ s<t}} \mathbb{E}\exp\left\{c_2\frac{\big|\gamma\big([0,s]\times[s,t]\big)\big|}{|t-s|^{1/2}}\right\} < \infty.$$

The proof is complete. $\square$

By theorem D.6 in Appendix and by the extension argument used for $\alpha(A)$, there is a continuous modification $\{\widetilde{\gamma}([0,t]^2_<)\}$ of the random family $\{\gamma([0,t]^2_<)\}$, such that for any $T>0$ there is a $c=c(T)>0$ such that

$$\mathbb{E}\exp\Big\{c\sup_{0\le s,t\le T}\big|\widetilde{\gamma}([0,t]^2_<)-\widetilde{\gamma}([0,s]^2_<)\big|\Big\}<\infty. \tag{2.4.12}$$

Notice that $\gamma([0,s]^2_<)=0$ as $s=0$, in particular,

$$\mathbb{E}\exp\Big\{c\sup_{0\le t\le T}\big|\widetilde{\gamma}([0,t]^2_<)\big|\Big\}<\infty. \tag{2.4.13}$$

For the sake of simplicity, from now on we use $\gamma([0,t]^2_<)$, rather than $\widetilde{\gamma}([0,t]^2_<)$, to denote this continuous process. And we call it *renormalized self-intersection local time* generated by the planar Brownian motion $W(t)$.

2.5 Notes and comments

Section 2.2.

We refer the reader to the book by Lawler ([109]) and the long monograph notes by Le Gall ([122]) for the general information on the intersections and self-intersections of Brownian motions and random walks. In addition to the historic works by Dvoretzsky, Erdös and Kakutani ([64], [65]), we mention the survey paper by Khoshnevisan ([103]) for an elementary proof of Theorem 2.2.1.

The first work on the intersection local times of independent Brownian motions is the paper by Wolpert ([161]). Dynkin ([66]) developed a way to construct additive functionals of several independent Markov processes, which includes Brownian intersection local times as a special case. In [67], Dynkin treated the special case of Brownian motions. Our approach of constructing $\alpha(A)$ is modified from Le Gall's work [122]. This treatment also leads to Le Gall's moment identity which plays a central role in Chapter 3.

A different method known as local non-determinism can be found in Geman and Horowitz [81] and in Geman, Horowitz and Rosen [82], where the idea is to view intersection local time as a local time of certain random field, and where the basic tool is Fourier analysis. We design the following exercise based on their idea.

Exercise 1. The following way of constructing the intersection local time $\alpha(A)$ constitutes the notation introduced in (2.1.2): For each $A \subset (\mathbb{R}^+)^p$, define the occupation measure $\mu_A(\cdot)$ on $\mathbb{R}^{d(p-1)}$ given by

$$\mu_A(B) = \int_A 1_B\big(W_1(s_1) - W_2(s_2), \cdots, W_{p-1}(s_{p-1}) - W_p(s_p)\big) ds_1 \cdots ds_p$$

where $B \subset \mathbb{R}^{d(p-1)}$.

(1). Prove that when $p(d-2) < d$,

$$\begin{aligned}&\mathbb{P}\Big\{\mu_A(\cdot) \text{ is absolutely continuous with respect to the Lebesgue} \\ &\qquad \text{measure on } \mathbb{R}^{d(p-1)} \text{ for every measurable and bounded } A \subset (\mathbb{R}^+)^p\Big\} \\ &= 1\end{aligned}$$

Hint: Define the random measure $\mu(\cdot)$ on $\mathbb{R}^{d(p-1)}$ as

$$\begin{aligned}\mu(B) = &\int_0^\infty \cdots \int_0^\infty ds_1 \cdots ds_p e^{-(s_1+\cdots+s_p)} \\ &\times 1_B\big(W_1(s_1) - W_2(s_2), \cdots, W_{p-1}(s_{p-1}) - W_p(s_p)\big) \qquad B \subset \mathbb{R}^{d(p-1)}\end{aligned}$$

and reduce the problem to the proof of almost sure absolute continuity of $\mu(\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^{d(p-1)}$; then establish the identity

$$\int_{(\mathbb{R}^d)^{p-1}} \mathbb{E}|\widehat{\mu}(\lambda_1, \cdots, \lambda_{p-1})|^2 d\lambda_1 \cdots d\lambda_{p-1} = (2\pi)^{d(p-1)} \int_{\mathbb{R}^d} G^p(x) dx < \infty$$

where

$$\widehat{\mu}(\lambda_1, \cdots, \lambda_{p-1}) = \int_{(\mathbb{R}^d)^{p-1}} e^{i(\lambda_1 \cdot x_1 + \cdots + \lambda_{p-1} \cdot x_{p-1})} \mu(dx_1, \cdots dx_{p-1})$$

is the Fourier transform of μ. Finally, use Plancherel-Parseval theorem (Theorem B.3, Appendix) to complete the proof.

(2). Let $\alpha(x, A)$ be the related Radon-Nikodym derivative. It can be shown ([82]) that there is a version version of $\alpha(x, A)$ (still denoted by $\alpha(x, A)$) such that for each x, $\alpha(x, \cdot)$ is a random measure on $(\mathbb{R}+)^p$, and for each $A \subset (\mathbb{R}+)^p$, $\alpha(\cdot, A)$ is continuous on $\mathbb{R}^d$. We define $\alpha(A) = \alpha(0, A)$. Prove that this construction produces the same intersection local time as the one defined in section 2.2.

Section 2.3.

The existence and continuity of self-intersection local time of Brownian motion has been investigated by Rosen ([138], [139]), by Yor ([165], [166]) and by Le Gall ([122]). The study has been extended to the setting of diffusion processes by Rosen ([142]) and to the setting of Lévy processes by Le Gall ([119]). The method used here is due to Le Gall ([122]). By different method, Rosen ((2.15), [139]) estimated the moment of $\beta(I)$ for I in a class wider than one defined in (2.3.5), in the setting of fractional Brownian motions.

Exercise 2. We demonstrate that when $p = 2$, the construction of $\beta(A)$ can be greatly simplified.

1. Let $h(x)$ be a bounded function on $\mathbb{R}^d$ and let $W(t)$ and $\widetilde{W}(t)$ be two independent d-dimensional Brownian motions. Prove that for any $0 \le a < b \le c < d$,

$$\begin{aligned}&\iint_{\{[a,b]\times[c,d]} h\big(W(t) - W(s)\big)dsdt\\ &\stackrel{d}{=} \iint_{\{[c-b,\ c-a]\times[0,\ d-c]} h\big(\widetilde{W}(t) - W(s)\big)dsdt.\end{aligned}$$

2. Simplify the construction of the random measure $\beta(A)$ in the special case $p = 2$ and $d \le 3$.

Section 2.4.

The idea of renormalization goes back to the paper by Varadhan ([153]), where for the first time the method of triangular approximation was introduced for some related but different models. This brilliant idea was later adopted by Le Gall ([115], [122], [123]) to the setting of self-intersection local time ([122]). The method used in this section comes from [122] and [123].

An alternative way of renormalization is by Tanaka formula. We refer to Rosen and Yor ([140] , [163], [145]) for this approach. See the paper [68] by Dynkin for a different treatment.

The renormalization of the p-multiple self-intersection local times is far more complicated as $d = 2$ and $p \ge 3$ and is not included in this book. We refer interested readers to the papers by Rosen ([141]), Rosen and Yor ([145]), Le Gall ([122]) on this problem.

In the case $d \ge 3$, the self-intersection local time of a d-dimensional Brownian motion can not be renormalized in the same way as when $d = 2$. Indeed, Yor ([164]), Calais and Yor ([20]) have proved that random variables

$$\begin{cases} \big(\log(1/\epsilon)\big)^{-1/2}\Big\{\beta_\epsilon\big([0,1]^2_<\big) - \mathbb{E}\beta_\epsilon\big([0,1]^2_<\big)\Big\} & \text{if } d = 3\\[2ex] \epsilon^{\frac{d-3}{2}}\Big\{\beta_\epsilon\big([0,1]^2_<\big) - \mathbb{E}\beta_\epsilon\big([0,1]^2_<\big)\Big\} & \text{if } d \ge 4\end{cases}$$

weakly converge to symmetric normal distributions as $\epsilon \to 0^+$. In particular, the blow-up of the renormalized quantity

$$\beta_\epsilon\big([0,1]^2_<\big) - \mathbb{E}\beta_\epsilon\big([0,1]^2_<\big)$$

(as $\epsilon \to 0^+$) shows that the renormalization fails in the case $d \geq 3$. In addition, Calais and Yor's observation links to the central limit theorem stated later in Theorem 5.5.3 in a natural way. See also the paper [92] by Hu and Nualart for extension of Calais and Yor' work to the setting of fractional Brownian motions.

Chapter 3

Large deviations: mutual intersection

-

3.1 High moment asymptotics

One of the major goal of this chapter is to provide precise estimate to the tail probability

$$\mathbb{P}\Big\{\alpha\big([0,1]^p\big) \geq t\Big\}$$

as $t \to \infty$. By Theorem 1.2.7, the problem is in connection to the study of the *high moment asymptotics* posted as

$$\mathbb{E}\Big[\alpha([0,1]^p)^m\Big] \qquad (m \to \infty).$$

By comparing (2.2.7) and (2.2.18), it seems that the the high moment asymptotics*high moment asymptotics* for

$$\mathbb{E}\Big[\alpha\big([0,\tau_1] \times \cdots \times [0,\tau_p]\big)^m\Big]$$

is more tractable. The question is: what can we say about the quantity

$$\int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \bigg[\sum_{\sigma\in\Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\bigg]^p$$

when $m \to \infty$?

For future reference, we generalize this problem to the study of

$$\int_{E^m} \pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p$$

where $(E,\mathcal{E},\pi)$ is a measure space, $K(x,y)$ is a non-negative measurable function on $E\times E$ and $x_{\sigma(0)}=x_0$ is an arbitrary but fixed point in E.

Theorem 3.1.1 *Let $p\ge 2$ be fixed. Assume that the kernel satisfies*

(1). Irreducibility: $K(x,y)>0$ for every $(x,y)\in E\times E$.

(2). Symmetry: $K(x,y)=K(y,x)$ for every $(x,y)\in E\times E$.

(3). Integrability:

$$\varrho\equiv\sup_f\iint_{E\times E}K(x,y)f(x)f(y)\pi(dx)\pi(dy)<\infty$$

where the supremum is taken over all f satisfying

$$\int_E |f(x)|^{\frac{2p}{2p-1}}\pi(dx)=1.$$

Then

$$\liminf_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\int_{E^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p \tag{3.1.1}$$

$$\ge p\log\varrho.$$

Proof. Let $f\in\mathcal{L}^{\frac{2p}{2p-1}}(E,\mathcal{E},\pi)$ be a bounded and non-negative function such that

$$\delta\equiv\inf_{x;\,f(x)>0}G(x_0,x)>0 \tag{3.1.2}$$

and that

$$\int_E f^{\frac{2p}{2p-1}}(x)\pi(dx)=1.$$

Let $g(x)=f^{\frac{2(p-1)}{2p-1}}(x)$ and $h(x)=f^{\frac{p}{2p-1}}(x)$. Then $f(x)=\sqrt{g(x)}h(x)$ and

$$\int_E g^{\frac{p}{p-1}}(x)\pi(dx)=\int_E h^2(x)\pi(dx)=1.$$

By boundedness of $f(x)$, there is a $\epsilon > 0$ such that $g(x) \geq \epsilon h^2(x)$. By Hölder inequality,

$$\begin{aligned}
&\left\{ \int_{E^m} \pi(dx_1) \cdots \pi(dx_m) \left[\sum_{\sigma \in \Sigma_m} \prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)}) \right]^p \right\}^{1/p} \qquad (3.1.3) \\
&\geq \int_{E^m} \pi(dx_1) \cdots \pi(dx_m) \Big(\prod_{k=1}^m g(x_k) \Big) \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m K((x_{\sigma(k-1)}, x_{\sigma(k)}) \\
&= m! \int_{E^m} \pi(dx_1) \cdots \pi(dx_m) \Big(\prod_{k=1}^m g(x_k) \Big) \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m K((x_{k-1}, x_k) \\
&= m! \int_{E^m} \pi(dx_1) \cdots \pi(dx_m) G(x_0, x_1) \sqrt{g(x_1)} \\
&\times \left[\prod_{k=2}^m \sqrt{g(x_{k-1})} K((x_{k-1}, x_k) \sqrt{g(x_k)} \right] \sqrt{g(x_m)} \\
&\geq \delta \epsilon m! \int_{E^m} \pi(dx_1) \cdots \pi(dx_m) h(x_1) \\
&\times \left[\prod_{k=2}^m \sqrt{g(x_{k-1})} K((x_{k-1}, x_k) \sqrt{g(x_k)} \right] h(x_m) \\
&= \delta \epsilon m! \langle h, T^{m-1} h \rangle
\end{aligned}$$

where the linear operator T: $\mathcal{L}^2(E, \mathcal{E}, \pi) \longrightarrow \mathcal{L}^2(E, \mathcal{E}, \pi)$ is defined as

$$(T\tilde{h})(x) = \sqrt{g(x)} \int_E K(x,y) \sqrt{g(y)} \tilde{h}(y) \pi(dy) \qquad \tilde{h} \in \mathcal{L}^2(E, \mathcal{E}, \pi).$$

For any $h_1, h_2 \in \mathcal{L}^2(E, \mathcal{E}, \pi)$,

$$\begin{aligned}
\langle h_1, Th_2 \rangle &= \iint_{E \times E} K(x,y) \sqrt{g(x)} h_1(x) \sqrt{g(y)} h_2(y) \pi(dx) \pi(dy) \\
&= \iint_{E \times E} K(y,x) \sqrt{g(x)} h_1(x) \sqrt{g(y)} h_2(y) \pi(dx) \pi(dy) = \langle Th_1, h_2 \rangle.
\end{aligned}$$

That is, T is symmetric. In addition,

$$\langle h_1, Th_2 \rangle = \frac{1}{4} \Big\{ \langle h_1 + h_2, T(h_1 + h_2) \rangle - \langle h_1 - h_2, T(h_1 - h_2) \rangle \Big\}.$$

Notice that

$$\begin{aligned}
&\Big| \langle h_1 \pm h_2, T(h_1 \pm h_2) \rangle \Big| \\
&\leq \iint_{E \times E} K(x,y) \sqrt{g(x)} \big| h_1(x) \pm h_2(x) \big| \\
&\times \sqrt{g(y)} \big| h_1(y) \pm h_2(y) \big| \pi(dx) \pi(dy).
\end{aligned}$$

By Hölder inequality,

$$\begin{aligned}
&\int_E \left|\sqrt{g(x)}\big(h_1(x) \pm h_2(x)\big)\right|^{\frac{2p}{2p-1}} \pi(dx) \\
&\le \left\{ \int_E g^{\frac{p}{p-1}}(x)\pi(dx) \right\}^{\frac{p-1}{2p-1}} \left\{ \int_E |h_1(x) \pm h_2(x)|^2 \pi(dx) \right\}^{\frac{p}{2p-1}} \\
&= \left\{ \int_E f^{\frac{2p}{2p-1}}(x)\pi(dx) \right\}^{\frac{p-1}{2p-1}} \left\{ \int_E |h_1(x) \pm h_2(x)|^2 \pi(dx) \right\}^{\frac{p}{2p-1}} \\
&= \left\{ \int_E |h_1(x) \pm h_2(x)|^2 \pi(dx) \right\}^{\frac{p}{2p-1}}.
\end{aligned}$$

Consequently,

$$\Big|\langle h_1 \pm h_2, T(h_1 \pm h_2)\rangle\Big| \le \varrho \int_E |h_1(x) \pm h_2(x)|^2 \pi(dx).$$

Hence,

$$\langle h_1, Th_2\rangle \le \frac{1}{4}\varrho\Big\{||h_1 + h_2||^2 + ||h_1 - h_2||^2\Big\} = \varrho||h_1|| \cdot ||h_2||.$$

Therefore, T is a bounded linear operator. This, together with symmetry, implies that T is self-adjoint.

By Theorem E.1 in Appendix, the self-adjoint operator T admits the spectral integral representation

$$T = \int_{-\infty}^{\infty} \lambda E(d\lambda). \tag{3.1.4}$$

By Corollary E.4 in Appendix,

$$T^{m-1} = \int_{-\infty}^{\infty} \lambda^{m-1} E(d\lambda) \tag{3.1.5}$$

By (E.15) in Appendix, the above representations lead to

$$\langle h, Th\rangle = \int_{-\infty}^{\infty} \lambda \mu_h(d\lambda) \tag{3.1.6}$$

$$\langle h, T^{m-1}h\rangle = \int_{-\infty}^{\infty} \lambda \mu_h(d\lambda) \tag{3.1.7}$$

where μ_h is a measure on $\mathbb{R}$ with

$$\mu_g(\mathbb{R}) = \int_E h^2(x)\pi(dx) = 1$$

(i.e., μ_h is a probability measure).

When m is odd (so $m-1$ is even), by Jensen inequality,

$$\begin{aligned}
&\langle h, T^{m-1}h\rangle \geq \left(\int_{-\infty}^{\infty} \lambda\mu_h(d\lambda)\right)^{m-1} = \Big(\langle h, Th\rangle\Big)^{m-1} \\
&= \left(\iint_{E\times E} K(x,y)\sqrt{g(x)}h(x)\sqrt{g(y)}h(y)\pi(dx)\pi(dy)\right)^{m-1} \\
&= \left(\iint_{E\times E} K(x,y)f(x)f(y)\pi(dx)\pi(dy)\right)^{m-1}.
\end{aligned}$$

Summarizing our argument, when m is odd,

$$\begin{aligned}
&\int_{E^m} \pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K((x_{\sigma(k-1)}, x_{\sigma(k)})\Big]^p \\
&\geq (\delta\epsilon)^p(m!)^p\left(\iint_{E\times E} K(x,y)f(x)f(y)\pi(dx)\pi(dy)\right)^{p(m-1)}.
\end{aligned}$$

When m is even, a slight modification of (3.1.3) gives

$$\begin{aligned}
&\left\{\int_{E^m} \pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\Big]^p\right\}^{1/p} \\
&\geq c(\delta,\epsilon)m!\langle h, T^{m-2}h\rangle
\end{aligned}$$

for some $c(\delta,\epsilon) > 0$. So the argument based on spectral representation leads to the lower bound

$$\begin{aligned}
&\int_{E^m} \pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K((x_{\sigma(k-1)}, x_{\sigma(k)})\Big]^p \\
&\geq c^p(\delta,\epsilon)(m!)^p\left(\iint_{E\times E} K(x,y)f(x)f(y)\pi(dx)\pi(dy)\right)^{p(m-2)}.
\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}
&\liminf_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\int_{E^m} \pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\Big]^p \\
&\geq p\log\iint_{E\times E} K(x,y)f(x)f(y)\pi(dx)\pi(dy).
\end{aligned}$$

Finally, the desired conclusion follows from the following two facts:

First, the supremum in the definition of ϱ can be taken among the non-negative f.

Second, by irreducibility assumption the set of functions f satisfying (3.1.2) is dense in non-negative function in $\mathcal{L}^{\frac{2p}{2p-1}}(E)$. □

The upper bound is much more difficult. Indeed, we can establish the upper bound only in the case when E is a finite set.

Theorem 3.1.2 *Fix $p \ge 2$, let E be a finite set and let $K\colon E \times E \longrightarrow \mathbb{R}^+$ be non-negative function such that $K(x,y) = K(y,x)$ for any $x, y \in E$. Let π be a non-negative function on E. Then*

$$\limsup_{m\to\infty} \frac{1}{m} \log \frac{1}{(m!)^p} \sum_{x_1,\cdots,x_m\in E} \pi(x_1)\cdots\pi(x_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\Big]^p \tag{3.1.8}$$

$$\le p \log \sup_f \sum_{x,y\in E} K(x,y) f(x) f(y) \pi(x)\pi(y)$$

where the supremum is taken over all functions f on Ω satisfying

$$\sum_{x\in E} |f(x)|^{\frac{2p}{2p-1}} \pi(x) = 1.$$

Proof. We may assume that $\pi(x) > 0$ for every $x \in E$, for otherwise we can remove all zero points of π from E.Let

$$\mu = L_m^{\mathbf{x}} = \frac{1}{m}\sum_{k=1}^m \delta_{x_k}$$

be the empirical measure generated by $\mathbf{x} = (x_1, \cdots, x_m)$. Notice that for each $\sigma \in \Sigma_m$

$$\sum_{y_1,\cdots,y_m\in E} 1_{\{L_m^{\mathbf{y}}=\mu\}} 1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}} = 1.$$

We have

$$\begin{aligned}
&\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\\
&= \sum_{y_1,\cdots,y_m\in E} 1_{\{L_m^{\mathbf{y}}=\mu\}} \sum_{\sigma\in\Sigma_m} 1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}} \prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\\
&= \sum_{y_1,\cdots,y_m\in E} 1_{\{L_m^{\mathbf{y}}=\mu\}} \prod_{k=1}^m K(y_{k-1}, y_k) \sum_{\sigma\in\Sigma_m} 1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}}.
\end{aligned}$$

We claim that

$$\sum_{\sigma\in\Sigma_m} 1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}} = \prod_{x\in E}\big(m\mu(x)\big)!.$$

Indeed, for each $x\in E$ there are, respectively, exactly $m\mu(x)$ of $x_1,\cdots,x_m$ and exactly $m\mu(x)$ of $y_1,\cdots,y_m$ which are equal to x. Therefore, there are $\big(m\mu(x)\big)!$ ways to match each x-valued component of $\mathbf{y}$ to an each x-valued component of $\mathbf{x}$. Thus, the claim follows from multiplication principle.

Consequently,

$$\begin{aligned}&\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\\&=\prod_{x\in E}\big(m\mu(x)\big)!\sum_{y_1,\cdots,y_m\in E}1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k).\end{aligned}$$

Let $q>1$ be the conjugate number of p and define $\varphi_\mu(x)=\mu(x)^{1/q}\pi(x)^{1/p}$. Then

$$\begin{aligned}&\sum_{y_1,\cdots,y_m\in E}\varphi_\mu(y_1)\cdots\varphi_\mu(y_m)\prod_{k=1}^m K(y_{k-1},y_k)\\&\ge\sum_{y_1,\cdots,y_m\in\Omega}\varphi_\mu(x_1)\cdots\varphi_\mu(x_m)1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k)\\&=\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)\sum_{y_1,\cdots,y_m\in E}1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k)\end{aligned}$$

where the last step follows from the fact that when $L_m^{\mathbf{y}}=\mu$, there are $m\mu(x)$ factors in the product $\varphi_\mu(x_1)\cdots\varphi_\mu(x_m)$ which are equal to $\varphi_\mu(x)$ for any $x\in E$.

Summarizing above steps,

$$\begin{aligned}&\frac{1}{(m!)^p}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\\&\le\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\frac{1}{m!}\Big(\prod_{x\in E}\big(m\mu(x)\big)!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\Big]^p\\&\times\Big[\sum_{y_1,\cdots,y_m\in E}\varphi_\mu(y_1)\cdots\varphi_\mu(y_m)\prod_{k=1}^m K(y_{k-1},y_k)\Big]^p.\end{aligned}$$

Define $g_\mu(x)=\mu(x)^{1/q}\pi(x)^{-1/q}$. Then $\varphi_\mu(x)=g_\mu(x)\pi(x)$ for every $x\in$

E and

$$\sum_{x\in E} g_\mu^q(x)\pi(x) = \sum_{x\in E}\mu(x) = 1, \qquad \sup_{x\in E} g_\mu(x) \le \sup_{x\in E}\pi(x)^{-1/q}(x).$$

Consequently,

$$\begin{aligned}
&\frac{1}{(m!)^p}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p \qquad (3.1.9)\\
&\le \sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\Big[\frac{1}{m!}\Big(\prod_{x\in E}(m\mu(x))!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\Big]^p\\
&\times\Big[\sup_g\sum_{y_1,\cdots,y_m\in E}\pi(y_1)\cdots\pi(y_m)\Big(\prod_{k=1}^m g(y_k)\Big)\prod_{k=1}^m K(y_{k-1},y_k)\Big]^p
\end{aligned}$$

where the supremum on the right hand side is taken for all non-negative functions g on E satisfying

$$\sum_{x\in E} g^q(x)\pi(x) = 1, \qquad \sup_{x\in E} g(x)\le c$$

and where $c = \sup_{x\in E}\pi(x)^{-1/q}(x)$.

For each g,

$$\begin{aligned}
&\sum_{y_1,\cdots,y_m\in E}\pi(y_1)\cdots\pi(y_m)\Big(\prod_{k=1}^m g(y_k)\Big)\prod_{k=1}^m K(y_{k-1},y_k)\\
&=\sum_{y_1,\cdots,y_m\in E}\pi(y_1)\cdots\pi(y_m)K(x_0,y_1)\sqrt{g(y_1)}\\
&\times\Big[\sqrt{g(y_{k-1})}\prod_{k=2}^m K(y_{k-1},y_k)\sqrt{g(x_k)}\Big]\sqrt{g(x_m)}\\
&\le c\big(\sup_{x,y\in E}K(x,y)\big)\Big(\sum_{x\in E}\pi(x)\Big)\sum_{y_1,\cdots,y_m\in E}\pi(y_1)\cdots\pi(y_m)h_o(y_1)\\
&\times\Big[\sqrt{g(y_{k-1})}\prod_{k=2}^m K(y_{k-1},y_k)\sqrt{g(x_k)}\Big]h_o(x_m)\\
&= c\big(\sup_{x,y\in E}K(x,y)\big)\Big(\sum_{x\in E}\pi(x)\Big)\langle h_o, T^{m-1}h_o\rangle
\end{aligned}$$

where $h_o(y)\equiv\Big(\sum_{x\in E}\pi(x)\Big)^{-1/2}$ on E and the bounded self-adjoint linear operator T: $\mathcal{L}^2(E,\mathcal{E},\pi)\longrightarrow\mathcal{L}^2(E,\mathcal{E},\pi)$ is defined by

$$(Th)(x) = \sqrt{g(x)}\sum_{y\in E}K(x,y)\sqrt{g(y)}h(y)\pi(y) \qquad h\in\mathcal{L}^2(E,\mathcal{E},\pi).$$

Similar to (3.1.7),

$$\langle h_o, T^{m-1}h_o\rangle = \int_{-\infty}^{\infty} \lambda^{m-1}\mu_{h_o}(d\lambda).$$

By the fact that $||h_0|| = 1$, μ_{h_o} is a probability measure on $\mathbb{R}$. By Theorem E.2 in Appendix, μ_{h_0} is supported on the interval

$$\Big[\inf_{||h||=1}\langle h, Th\rangle, \quad \sup_{||h||=1}\langle h, Th\rangle\Big].$$

Thus,

$$\langle h_o, T^{m-1}h_o\rangle \le \left(\max\left\{\Big|\inf_{||h||=1}\langle h, Th\rangle\Big|, \ \Big|\sup_{||h||=1}\langle h, Th\rangle\Big|\right\}\right)^{m-1}$$
$$= \sup_{||h||=1}\left|\sum_{x,y\in E} K(x,y)\sqrt{g(x)}h(x)\sqrt{g(y)}h(y)\pi(x)\pi(y)\right|^{m-1}.$$

Write $f(x) = \sqrt{g(x)}h(x)$. Then $f(x) \ge 0$ and

$$\sum_{x\in E} f^{\frac{2p}{2p-1}}(x)\pi(x) \le \left\{\sum_{x\in E} g^q(x)\pi(x)\right\}^{\frac{p-1}{2p-1}}\left\{\sum_{x\in E} h^2(x)\pi(x)\right\}^{\frac{p}{2p-1}} = 1.$$

Consequently,

$$\langle h_o, T^{m-1}h_o\rangle \le \varrho^{m-1}$$

where

$$\varrho = \sup\left\{\sum_{x,y\in E} K(x,y)f(x)f(y)\pi(x)\pi(y); \quad \sum_{x\in E}|f(x)|^{\frac{2p}{2p-1}}\pi(x) = 1\right\}.$$

Summarizing our argument,

$$\limsup_{m\to\infty}\frac{1}{m}\log\sup_g \sum_{y_1,\cdots,y_m\in E}\pi(y_1)\cdots\pi(y_m) \qquad (3.1.10)$$
$$\times\Big(\prod_{k=1}^m g(y_k)\Big)\prod_{k=1}^m K(y_{k-1},y_k) \le \log\varrho.$$

In view of (3.1.9), it remains to show

$$\limsup_{m\to\infty}\frac{1}{m}\log\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m) \qquad (3.1.11)$$
$$\times\left[\frac{1}{m!}\Big(\prod_{x\in E}(m\mu(x))!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\right]^p \le 0.$$

Let $\mathcal{P}_m(E)$ be the set of the probability measures ν on A such that for each $x \in E$, $\nu(x) = k/n$ for some integer $0 \le k \le m$.

$$
\begin{aligned}
&\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\bigg[\frac{1}{m!}\Big(\prod_{x\in E}\big(m\mu(x)\big)!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\bigg]^p\\
&=\sum_{\nu\in\mathcal{P}_m(E)}1_{\{\mu=\nu\}}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\\
&\times\bigg[\frac{1}{m!}\Big(\prod_{x\in E}\big(m\nu(x)\big)!\Big)\Big(\prod_{x\in E}\varphi_\nu(x)^{m\nu(x)}\Big)^{-1}\bigg]^p\\
&=\sum_{\nu\in\mathcal{P}_m(E)}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\\
&\times\bigg[\frac{1}{m!}\Big(\prod_{x\in E}\big(m\nu(x)\big)!\Big)\Big(\prod_{x\in E}\varphi_\nu(x)^{m\nu(x)}\Big)^{-1}\bigg]^p.
\end{aligned}
$$

By Stirling formula, $m! \sim \sqrt{2\pi m}m^m e^{-m}$ and

$$
\big(m\nu(x)\big)! \le C\sqrt{m\nu(x)}\big(m\nu(x)\big)^{m\nu(x)}e^{m\nu(x)} \qquad x \in E.
$$

Hence,

$$
\frac{1}{m!}\Big(\prod_{x\in E}\big(m\nu(x)\big)!\Big) \le Cm^{\#(E)/2}\prod_{x\in E}\nu(x)^{m\nu(x)}.
$$

Recall that $\varphi_\nu(x) = \nu(x)^{1/q}\pi(x)^{1/p}$. Therefore,

$$
\begin{aligned}
&\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\bigg[\frac{1}{m!}\Big(\prod_{x\in E}\big(m\nu(x)\big)!\Big)\Big(\prod_{x\in E}\varphi_\nu(x)^{m\nu(x)}\Big)^{-1}\bigg]^p\\
&\le C^p m^{p\#(E)/2}\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\prod_{x\in E}\Big(\frac{\nu(x)}{\pi(x)}\Big)^{m\nu(x)}\\
&= C^p m^{p\#(E)/2}\sum_{x_1,\cdots,x_m\in E}\nu(x_1)\cdots\nu(x_m) = C^p m^{p\#(E)/2}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\sum_{x_1,\cdots,x_m\in E}\pi(x_1)\cdots\pi(x_m)\bigg[\frac{1}{m!}\Big(\prod_{x\in E}\big(m\mu(x)\big)!\Big)\Big(\prod_{x\in E}\varphi_\mu(x)^{m\mu(x)}\Big)^{-1}\bigg]^p\\
&\le C^p m^{p\#(E)/2}\#\{\mathcal{P}_m(E)\}.
\end{aligned}
$$

Finally, (3.1.11) follows from the fact that $\#\{\mathcal{P}_m(E)\}$ is equal to the number of the non-negative lattice solutions $(k(x);\ \ x \in E)$ of the equation

$$
\sum_{x\in E}k(x) = m.
$$

The later is equal to

$$\binom{m+\#(E)-1}{\#(E)-1}.$$

$\square$

3.2 High moment of $\alpha([0,\tau_1]\times\cdots\times[0,\tau_p])$

We now return to the intersection local time $\alpha(A)$ of independent d-dimensional Brownian motions $W_1(t),\cdots,W_p(t)$. Throughout we assume that $p(d-2)<d$.

Define

$$\rho=\sup\left\{\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(x-y)f(x)f(y)dxdy;\quad\int_{\mathbb{R}^d}|f(x)|^{\frac{2p}{2p-1}}dx=1\right\} \quad (3.2.1)$$

Lemma 3.2.1 . *Under $p(d-2)<d$,*

$$0<\rho\le\left(\int_{\mathbb{R}^d}G^p(x)dx\right)^{1/p}<\infty.$$

Proof. The lower bound $\rho>0$ is obvious. We now prove upper bound. We may only consider non-negative f. By Hölder inequality,

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(x-y)f(x)f(y)dxdy\\
&=\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(x-y)\big(f(x)f(y)\big)^{\frac{1}{2p-1}}\big(f(x)f(y)\big)^{\frac{2(p-1)}{2p-1}}dxdy\\
&\le\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}G^p(x-y)\big(f(x)f(y)\big)^{\frac{p}{2p-1}}dxdy\right)^{1/p}\\
&\times\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}\big(f(x)f(y)\big)^{\frac{2p}{2p-1}}dxdy\right)^{1/q}\\
&=\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}G^p(x-y)\big(f(x)f(y)\big)^{\frac{p}{2p-1}}dxdy\right)^{1/p}
\end{aligned}$$

where $q=p(p-1)^{-1}$ is the conjugate number of p. By translation invariance and Fubini theorem,

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d}G^p(x-y)\big(f(x)f(y)\big)^{\frac{p}{2p-1}}dxdy\\
&=\int_{\mathbb{R}^d}G^p(x)\left(\int_{\mathbb{R}^d}\big(f(x+y)f(y)\big)^{\frac{p}{2p-1}}dy\right)dx.
\end{aligned}$$

By Cauchy-Schwartz inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big(f(x+y)f(y)\big)^{\frac{p}{2p-1}} dy \\
&\le \left(\int_{\mathbb{R}^d} f(x+y)^{\frac{2p}{2p-1}} dy\right)^{1/2} \left(\int_{\mathbb{R}^d} f(y)^{\frac{2p}{2p-1}} dy\right)^{1/2} \\
&= \int_{\mathbb{R}^d} f(y)^{\frac{2p}{2p-1}} dy = 1.
\end{aligned}$$

where the second step partially follows from shifting invariance. Thus,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} G^p(x-y)\big(f(x)f(y)\big)^{\frac{p}{2p-1}} dxdy \le \int_{\mathbb{R}^d} G^p(x)dx.$$

□

Recall that $\tau_1, \cdots, \tau_p$ are i.i.d. exponential times of parameter 1 and that $\tau_1, \cdots, \tau_p$ are independent of $W_1(t), \cdots, W_p(t)$. The following is the main result of this section.

Theorem 3.2.2 . *Under $p(d-2) < d$,*

$$\lim_{m\to\infty} \frac{1}{m} \log \frac{1}{(m!)^p} \mathbb{E}\Big[\alpha([0,\tau_1]\times\cdots\times[0,\tau_p])^m\Big] = p\log\rho.$$

Proof. By (2.2.18), the lower bound of (3.2.2) is a direct consequence of Theorem 3.1.1, and the upper bound is equivalent to

$$\begin{aligned}
&\limsup_{m\to\infty} \frac{1}{m} \log \frac{1}{(m!)^p} \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \\
&\times \Big[\sum_{\sigma\in\Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\Big]^p \le p\log\rho.
\end{aligned} \tag{3.2.2}$$

Based on Theorem 3.1.2, our approach relies on compactification and democratization. The fact (Theorem A.1, Appendix) that the function $G(x)$ is discontinuous at 0 when $d \ge 2$ makes matter delicate.

The compactification procedure depends on the folding method. Let $N >$

0 be large but fixed.

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p\\
&=\sum_{z_1,\cdots,z_m\in\mathbb{Z}^d}\int_{([-N,N]^d)^m} dy_1\cdots dy_m\\
&\times\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G\big((2Nz_{\sigma(k)}+y_{\sigma(k)})-(2Nz_{\sigma(k-1)}+y_{\sigma(k-1)})\big)\Big]^p\\
&=\int_{([-N,N]^d)^m} dy_1\cdots dy_m\sum_{z_1,\cdots,z_m\in\mathbb{Z}^d}\\
&\times\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G\big(2N(z_{\sigma(k)}-z_{\sigma(k-1)})+(y_{\sigma(k)}-y_{\sigma(k-1)})\big)\Big]^p.
\end{aligned}$$

Fix $y_1,\cdots,y_m\in[-N,N]^d$. By triangular inequality,

$$\begin{aligned}
&\Big\{\sum_{z_1,\cdots,z_m\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G\big(2N(z_{\sigma(k)}-z_{\sigma(k-1)})+(y_{\sigma(k)}-y_{\sigma(k-1)})\big)\Big]^p\Big\}^{1/p}\\
&\le\sum_{\sigma\in\Sigma_m}\Big\{\sum_{z_1,\cdots,z_m\in\mathbb{Z}^d}\prod_{k=1}^m G^p\big(2N(z_{\sigma(k)}-z_{\sigma(k-1)})+(y_{\sigma(k)}-y_{\sigma(k-1)})\big)\Big\}^{1/p}\\
&=\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G_N(y_{\sigma(k)}-y_{\sigma(k-1)}).
\end{aligned}$$

where

$$G_N(x)=\Big\{\sum_{z\in\mathbb{Z}^d}G^p(2Nz+x)\Big\}^{1/p}.$$

Clearly, $G_N(x)$ is symmetric and periodic:

$$G_N(x+2Ny)=G_N(x)\qquad x\in\mathbb{R}^d\quad y\in\mathbb{Z}^d$$

and

$$\int_{[-N,N]^d}G_N^p(x)dx=\sum_{z\in\mathbb{Z}^d}\int_{[-N,N]^d}G^p(2Nz+x)dx=\int_{\mathbb{R}^d}G^p(x)dx<\infty.$$

Summarizing what we have,

$$\int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\Big]^p \tag{3.2.3}$$
$$\leq \int_{([-N,N]^d)^m} dy_1 \cdots dy_m \Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G_N(y_{\sigma(k)} - y_{\sigma(k-1)})\Big]^p.$$

For integer $n \geq 1$, partition $[-N,N]^d$ uniformly into 2^{nd} small boxes with the side length $2\epsilon = N/2^n$. Let E be the set of the geometric centers of these small boxes. Write

$$Q_\epsilon = [-\epsilon, \epsilon]^d.$$

Then

$$\int_{([-N,N]^d)^m} dx_1 \cdots dx_m \Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G_N(x_{\sigma(k)} - x_{\sigma(k-1)})\Big]^p$$
$$= \sum_{y_1,\cdots,y_m\in E}\int_{Q_\epsilon^m} dz_1\cdots dz_m$$
$$\times \Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G_N\big((z_{\sigma(k)} - z_{\sigma(k-1)}) + (y_{\sigma(k)} - y_{\sigma(k-1)})\big)\Big]^p.$$

By triangular inequality,

$$\int_{Q_\epsilon^m} dz_1\cdots dz_m \Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G_N\big((z_{\sigma(k)} - z_{\sigma(k-1)}) + (y_{\sigma(k)} - y_{\sigma(k-1)})\big)\Big]^p$$
$$\leq \Big[\sum_{\sigma\in\Sigma_m}\Big\{\int_{Q_\epsilon^m} dz_1\cdots dz_m \prod_{k=1}^m G_N^p\big((z_k - z_{k-1}) + (y_{\sigma(k)} - y_{\sigma(k-1)})\big)\Big\}^{1/p}\Big]^p$$
$$\leq (2\epsilon)^{md}\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G_{N,\epsilon}(y_{\sigma(k)} - y_{\sigma(k-1)})\Big]^p$$

where

$$G_{N,\epsilon}(x) = \Big\{\frac{1}{(2\epsilon)^d}\sup_{y\in Q_\epsilon}\int_{Q_\epsilon} G_N^p(x+y+z)dz\Big\}^{1/p}.$$

Clearly, $G_{N,\epsilon}(x)$ is symmetric and enjoys the periodicity described by

$$G_{N,\epsilon}(x + 2Ny) = G_{N,\epsilon}(x) \qquad x\in\mathbb{R}^d \quad y\in\mathbb{Z}^d.$$

Summarizing our discussion since (3.2.3),

$$\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p$$
$$\le (2\epsilon)^{md}\sum_{y_1,\cdots,y_m\in E}\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G_{N,\epsilon}(y_{\sigma(k)}-y_{\sigma(k-1)})\Big]^p.$$

Applying Theorem 3.1.2 (with $\pi(x)\equiv(2\epsilon)^d$) to the right hand side gives,

$$\limsup_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p$$
$$\le p\log\rho_{N,\epsilon}$$

where

$$\rho_{N,\epsilon}=\Big\{(2\epsilon)^{2d}\sum_{x,y\in E}G_{N,\epsilon}(x-y)f(x)f(y);\qquad(3.2.4)$$
$$\sum_{x\in E}|f(x)|^{\frac{2p}{2p-1}}(2\epsilon)^d=1\Big\}.$$

Finally, (3.2.2) follows from the following Lemma 3.2.3 and Lemma 3.2.4. □

Lemma 3.2.3 *Let $\rho_{N,\epsilon}$ be defined in (3.2.4).*

$$\limsup_{\epsilon\to0^+}\rho_{N,\epsilon}\le\rho_N$$

where

$$\rho_N=\Big\{\iint_{\{[-N,N]^d\times[-N,N]^d\}}G_N(u-v)f(u)f(v)dudv;\qquad(3.2.5)$$
$$\int_{[-N,N]^d}|f(u)|^{\frac{2p}{2p-1}}du=1\Big\}.$$

Proof. For fixed n_0 write $2\delta=N/2^{n_0}$. Set

$$G_N^{(1)}(x)=G_N(x)1_{\{x\notin Q_\delta\}},\qquad G_N^{(2)}(x)=G_N(x)1_{\{x\in Q_\delta\}}.$$

Then $G_N^{(1)}$ and $G_N^{(2)}$ are symmetric and periodic. By triangular inequality

$$G_{N,\epsilon}(x)\le G_{N,\epsilon}^{(1)}(x)+G_{N,\epsilon}^{(2)}(x)$$

where

$$G_{N,\epsilon}^{(i)}(x) = \left\{ \frac{1}{(2\epsilon)^d} \sup_{y\in Q_\epsilon} \int_{Q_\epsilon} \left[G_{N,\epsilon}^{(i)}(x+y+z)\right]^p dz \right\}^{1/p} \qquad i=1,2.$$

Consequently,

$$\rho_{N,\epsilon} \le \rho_{N,\epsilon}^{(1)} + \rho_{N,\epsilon}^{(2)} \tag{3.2.6}$$

where

$$\rho_{N,\epsilon}^{(i)} = \left\{ (2\epsilon)^{2d} \sum_{x,y\in E} G_{N,\epsilon}^{(i)}(x-y) f(x) f(y);\quad \sum_{x\in E} |f(x)|^{\frac{2p}{2p-1}} (2\epsilon)^d = 1 \right\}$$

$i = 1, 2$.

By modifying the argument in the proof of Lemma 3.1.1, one can prove

$$\rho_{N,\epsilon}^{(1)} \le (2\epsilon)^d \left\{ \sum_{x\in E} \left[G_{N,\epsilon}^{(1)}(x)\right]^p \right\}^{1/p}.$$

Indeed, the periodicity of the function $G_{N,\epsilon}^{(1)}(x)$ allows the argument by translation invariance carried out in the proof of Lemma 3.1.1.

By the definition of $G_{N,\epsilon}^{(1)}(x)$, therefore,

$$\rho_{N,\epsilon}^{(1)} \le \left\{ \sum_{x\in E} \sup_{y\in Q_\epsilon} \int_{Q_\epsilon} \left[G_N^{(1)}(x+y+z)\right]^p dz \right\}^{1/p}.$$

Notice that for any $y \in Q_\epsilon$,

$$\begin{aligned} &\int_{Q_\epsilon} \left[G_N^{(1)}(x+y+z)\right]^p dz = \int_{y+Q_\epsilon} \left[G_N^{(1)}(x+z)\right]^p dz \\ &\le \int_{Q_{2\epsilon}} \left[G_N^{(1)}(x+z)\right]^p dz = \int_{x+Q_{2\epsilon}} \left[G_N^{(1)}(z)\right]^p dz. \end{aligned}$$

Observe that the family $\{x+Q_{2\epsilon},\quad x\in E\}$ is contained in $[-N-\epsilon, N+\epsilon]^d$ and that every point in $[-N-\epsilon, N+\epsilon]^d$ is covered by this family at most 2^d times. Thus

$$\rho_{N,\epsilon}^{(1)} \le \left\{ \sum_{x\in E} \int_{x+Q_{2\epsilon}} \left[G_N^{(1)}(z)\right]^p dz \right\}^{1/p} \tag{3.2.7}$$

$$\le \left\{ 2^d \int_{[-N-\epsilon,N+\epsilon]^d} \left[G_N^{(1)}(z)\right]^p dz \right\}^{1/p} = 2^{d/p} \left\{ \int_{Q_\delta} G_N^p(x) dx \right\}^{1/p}.$$

By the fact that $G_N(x)$ is continuous and bounded away from 0, there is a constant $K>0$ such that $G_N^{(2)}(x)\le K$ for any $x\in\mathbb{R}^d$. Write $G^*(x)=\min\{G_N(x),K\}$. Then $G^*(x)$ is a continuous, symmetric and periodic function such that $G_N^{(2)}\le G^*\le G_N$. In particular,

$$\rho^{(2)}_{N,\epsilon}\le\rho^*_\epsilon \tag{3.2.8}$$

where

$$\rho^*_\epsilon=\sup\left\{(2\epsilon)^{2d}\sum_{x,y\in E}G^*_\epsilon(x-y)f(x)f(y);\quad\sum_{x\in E}|f(x)|^{\frac{2p}{2p-1}}(2\epsilon)^d=1\right\}$$

and

$$G^*_\epsilon(x)=\left\{\frac{1}{(2\epsilon)^d}\sup_{y\in Q_\epsilon}\int_{Q_\epsilon}\big[G^*(x+y+z)\big]^p dz\right\}^{1/p}\qquad x\in\mathbb{R}^d.$$

By continuity of G^* one can easily establish

$$\lim_{\epsilon\to0^+}\int_{[-N,N]^d}\big|G^*_\epsilon(u)-G^*(u)\big|^p du=0.$$

By triangular inequality,

$$\Delta_\epsilon=\sup_{|u-v|\le2\epsilon}\big|G^*_\epsilon(u)-G^*_\epsilon(v)\big|\longrightarrow0\qquad(\epsilon\to0^+).$$

Fixed the non-negative function $f(x)$ on E with

$$\sum_{x\in E}f^{\frac{2p}{2p-1}}(x)(2\epsilon)^d=1.$$

We extend it into a function on $[-N,N]^d$ by letting $f(u)=f(x)$ for u in the small box centered by x. We have

$$\int_{[-N,N]^d}f^{\frac{2p}{2p-1}}(u)du=1$$

and

$$\begin{aligned}&(2\epsilon)^{2d}\sum_{x,y\in E}G^*_\epsilon(x-y)f(x)f(y)\\&\le\iint_{\{[-N,N]^d\times[-N,N]^d\}}G^*(u-v)f(u)f(v)dudv\\&+\iint_{\{[-N,N]^d\times[-N,N]^d\}}|G^*_\epsilon(u-v)-G^*(u-v)|f(u)f(v)dudv\\&+\Delta_\epsilon\iint_{\{[-N,N]^d\times[-N,N]^d\}}f(u)f(v)dudv.\end{aligned}$$

For the first term, by the relation $G^* \le G_N$

$$\iint_{\{[-N,N]^d\times[-N,N]^d\}} G^*(u-v)f(u)f(v)dudv \le \rho_N.$$

For the second term, a procedure similar to the proof of Lemma 3.1.1 leads to the conclusion that

$$\begin{aligned}&\iint_{\{[-N,N]^d\times[-N,N]^d\}} |G^*_\epsilon(u-v)-G^*(u-v)|f(u)f(v)dudv\\ &\le \left\{\int_{[-N,N]^d}\Big|G^*_\epsilon(u)-G^*(u)\Big|^p du\right\}^{1/p}.\end{aligned}$$

As for the third term,

$$\begin{aligned}&\iint_{\{[-N,N]^d\times[-N,N]^d\}} f(u)f(v)dudv = \left(\int_{[-N,N]^d} f(u)du\right)^2\\ &\le \Big((2N)^d\Big)^{1/p}\left(\int_{[-N,N]^d} f^{\frac{2p}{2p-1}}(u)du\right)^{\frac{2p-1}{p}} = \Big((2N)^d\Big)^{1/p}.\end{aligned}$$

Summarizing our argument,

$$\rho^*_\epsilon \le \rho_N + \left\{\int_{[-N,N]^d}\Big|G^*_\epsilon(u)-G^*(u)\Big|^p du\right\}^{1/p} + \Delta_\epsilon\Big((2N)^d\Big)^{1/p}.$$

By (3.2.8),

$$\limsup_{\epsilon\to 0^+}\rho^{(2)}_{N,\epsilon} \le \rho_N.$$

In view of (3.2.6) and (3.2.7),

$$\limsup_{\epsilon\to 0^+}\rho_{N,\epsilon} \le \rho_N + 2^{d/p}\left\{\int_{Q_\delta} G^p_N(x)dx\right\}^{1/p}.$$

Letting $\delta \to 0^+$ leads to the desired conclusion. □

Lemma 3.2.4 *Let ρ and ρ_N be defined as in (3.1.1) and (3.2.5), respectively.*

$$\limsup_{N\to\infty}\rho_N \le \rho. \tag{3.2.9}$$

Proof. Let $M < N$ be fixed and set

$$G^{(1)}(x) = G(x)1_{[-M,M]^d}(x), \qquad G^{(2)}(x) = G(x) - G_1(x).$$

By triangular inequality

$$G_N(x) \le G_N^{(1)}(x) + G_N^{(2)}(x) \qquad x\in\mathbb{R}^d$$

where

$$G_N^{(i)}(x) = \Big\{\sum_{z\in\mathbb{Z}^d}\big[G_N^{(i)}(2Nz+x)\big]^p\Big\}^{1/p} \qquad i=1,2.$$

Consequently,

$$\rho_N \le \rho_N^{(1)} + \rho_N^{(2)} \tag{3.2.10}$$

where

$$\rho_N^{(i)} = \sup\Big\{\iint_{\{[-N,N]^d\times[-N,N]^d\}} G_N^{(i)}(u-v)f(u)f(v)dudv;$$
$$\int_{[-N,N]^d}|f(u)|^{\frac{2p}{2p-1}}du = 1\Big\} \qquad i=1,2.$$

By a argument similar to the proof of Lemma 3.1.1,

$$\rho_N^{(2)} \le \Big\{\int_{[-N,N]^d}\big[G_N^{(2)}(x)\big]^p dx\Big\}^{1/p} \tag{3.2.11}$$
$$= \Big\{\int_{\{\mathbb{R}^d\setminus[-M,M]^d\}} G^p(x)dx\Big\}^{1/p}.$$

By definition, $G_N^{(1)}(x)$ is periodic and

$$G_N^{(1)}(x) = G^{(1)}(x) \qquad x\in[-2(N-M),2(N-M)]^d. \tag{3.2.12}$$

Write $E_N = [-N,N]^d\setminus[N-M,N-M]^d$ and let $f(x)\ge 0$ be a non-negative function on $[-N,N]^d$ such that

$$\int_{[-N,N]^d} f^{\frac{2p}{2p-1}}(x)dx = 1.$$

We extend f into a periodic function on $\mathbb{R}^d$ with

$$f(x+2Ny) = f(x) \qquad x\in\mathbb{R}^d \qquad y\in\mathbb{Z}^d.$$

We claim that there is $a \in [-N,N]^d$ such that

$$\int_{E_N} f^{\frac{2p}{2p-1}}(x+a)dx \le \frac{|E_N|}{(2N)^d} \tag{3.2.13}$$

where $|E_N|$ is the volume of E_N.

If not, we would have

$$\int_{[-N,N]^d}\int_{E_N} f^{\frac{2p}{2p-1}}(x+y)dxdy > \int_{[-N,N]^d} \frac{|E_N|}{(2N)^d}dy = |E_N|.$$

This can not be true since

$$\begin{aligned}&\int_{[-N,N]^d}\int_{E_N} f^{\frac{2p}{2p-1}}(x+y)dxdy\\ &= \int_{E_N}\int_{[-N,N]^d} f^{\frac{2p}{2p-1}}(x+y)dydx = \int_{E_N} dx = |E_N|.\end{aligned}$$

Define $\bar{f}(x) = f(x+a)$. Then

$$\int_{[-N,N]^d} \bar{f}^{\frac{2p}{2p-1}}(x)dx = \int_{[-N,N]^d} f^{\frac{2p}{2p-1}}(x)dx = 1.$$

Define

$$h(x,y) = G_N^{(1)}(x-y)f(x)f(y).$$

By the translation invariance led by periodicity,

$$\begin{aligned}&\iint_{[-N,N]^d\times[-N,N]^d} G_N^{(1)}(x-y)f(x)f(y)dxdy\\ &= \iint_{[-N,N]^d\times[-N,N]^d} h(x,y)dxdy\\ &= \iint_{[-N,N]^d\times[-N,N]^d} h(x+a,y+a)dxdy\\ &= \iint_{[-N,N]^d\times[-N,N]^d} G_N^{(1)}(x-y)\bar{f}(x)\bar{f}(y)dxdy\\ &\le \iint_{([-N,N]^d\setminus E_N)\times([-N,N]^d\setminus E_N)} G_N^{(1)}(x-y)\bar{f}(x)\bar{f}(y)dxdy\\ &+ 2\iint_{E_N\times[-N,N]^d} G_N^{(1)}(x-y)\bar{f}(x)\bar{f}(y)dxdy.\end{aligned}$$

Notice that as $(x,y)\in([-N,N]^d\setminus E_N)\times([-N,N]^d\setminus E_N)$, $x-y\in[-2(N-M),2(N-M)]^d$. By (3.2.12),

$$\begin{aligned}
&\iint_{([-N,N]^d\setminus E_N)\times([-N,N]^d\setminus E_N)} G_N^{(1)}(x-y)\bar f(x)\bar f(y)dxdy\\
&=\iint_{([-N,N]^d\setminus E_N)\times([-N,N]^d\setminus E_N)} G^{(1)}(x-y)\bar f(x)\bar f(y)dxdy\\
&\le\iint_{[-N,N]^d\times[-N,N]^d} G(x-y)\bar f(x)\bar f(y)dxdy\le\rho.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\iint_{E_N\times[-N,N]^d} G_N^{(1)}(x-y)\bar f(x)\bar f(y)dxdy\\
&=\iint_{E_N\times[-N,N]^d} G_N^{(1)}(x-y)\big(\bar f(x)\bar f(y)\big)^{\frac{1}{2p-1}}\big(\bar f(x)\bar f(y)\big)^{\frac{2(p-1)}{2p-1}}dxdy\\
&\le\bigg\{\iint_{[-N,N]^d\times[-N,N]^d}\big[G_N^{(1)}(x-y)\big]^p\big(\bar f(x)\bar f(y)\big)^{\frac{p}{2p-1}}dxdy\bigg\}^{1/p}\\
&\times\bigg\{\iint_{E_N\times[-N,N]^d}\big(\bar f(x)\bar f(y)\big)^{\frac{2p}{2p-1}}dxdy\bigg\}^{\frac{p-1}{p}}.
\end{aligned}$$

By (3.2.13)

$$\iint_{E_N\times[-N,N]^d}\big(\bar f(x)\bar f(y)\big)^{\frac{2p}{2p-1}}dxdy\le\frac{|E_N|}{(2N)^d}.$$

By periodicity

$$\begin{aligned}
&\iint_{[-N,N]^d\times[-N,N]^d}\big[G_N^{(1)}(x-y)\big]^p\big(\bar f(x)\bar f(y)\big)^{\frac{p}{2p-1}}dxdy\\
&=\iint_{[-N,N]^d\times[-N,N]^d}\big[G_N^{(1)}(x)\big]^p\big(\bar f(x+y)\bar f(y)\big)^{\frac{p}{2p-1}}dxdy\\
&=\int_{[-N,N]^d}\big[G_N^{(1)}(x)\big]^p\bigg[\int_{[-N,N]^d}\big(\bar f(x+y)\bar f(y)\big)^{\frac{p}{2p-1}}dy\bigg]dx.
\end{aligned}$$

By periodicity again,

$$\begin{aligned}
&\int_{[-N,N]^d}\big(\bar f(x+y)\bar f(y)\big)^{\frac{p}{2p-1}}dy\\
&\le\bigg\{\int_{[-N,N]^d}\bar f^{\frac{2p}{2p-1}}(x+y)dy\bigg\}^{1/2}\bigg\{\int_{[-N,N]^d}\bar f^{\frac{2p}{2p-1}}(y)dy\bigg\}^{1/2}=1.
\end{aligned}$$

Therefore,

$$\begin{aligned}&\iint_{[-N,N]^d\times[-N,N]^d}\big[G_N^{(1)}(x-y)\big]^p\big(\bar f(x)\bar f(y)\big)^{\frac{p}{2p-1}}dxdy\\&\le\int_{[-N,N]^d}G_N^p(x)dx=\int_{\mathbb{R}^d}\big[G^{(1)}(x)\big]^p dx\le\int_{\mathbb{R}^d}G^p(x)dx.\end{aligned}$$

Summarizing our argument,

$$\begin{aligned}&\iint_{[-N,N]^d\times[-N,N]^d}G_N^{(1)}(x-y)f(x)f(y)dxdy\\&\le\rho+2\Big(\frac{|E_N|}{(2N)^d}\Big)^{\frac{p-1}{p}}\Big\{\int_{\mathbb{R}^d}G^p(x)dx\Big\}^{1/p}.\end{aligned}$$

Taking supremum over f on the right hand side gives

$$\rho_N^{(1)}\le\rho+2\Big(\frac{|E_N|}{(2N)^d}\Big)^{\frac{p-1}{p}}\Big\{\int_{\mathbb{R}^d}G^p(x)dx\Big\}^{1/p}. \tag{3.2.14}$$

By the obvious fact

$$\frac{|E_N|}{(2N)^d}\longrightarrow 0\qquad(N\to\infty),$$

$$\limsup_{N\to\infty}\rho_N^{(1)}\le\rho.$$

In view of (3.2.10) and (3.2.11)

$$\limsup_{N\to\infty}\rho_N\le\rho+\Big\{\int_{\{\mathbb{R}^d\setminus[-M,M]^d\}}G^p(x)dx\Big\}^{1/p}.$$

Letting $M\to\infty$ on the right hand side yields the desired conclusion. $\square$

3.3 Large deviation for $\alpha\big([0,1]^p\big)$

By Theorem 1.2.7, an immediate consequence of Theorem 3.2.2 is the following result in large deviations.

Theorem 3.3.1 *Under $p(d-2)<d$,*

$$\lim_{t\to\infty}t^{-1/p}\log\mathbb{P}\Big\{\alpha([0,\tau_1]\times\cdots\times[0,\tau_p])\ge t\Big\}=-\frac{p}{\rho}$$

We are more interested in the tail of the intersection local time up to a deterministic time. In view of the scaling property given in Proposition 2.2.6, we may let the Brownian motions run up to the time 1.

Theorem 3.3.2 *Under $p(d-2)<d$,*

$$\lim_{t\to\infty} t^{-\frac{2}{d(p-1)}}\log\mathbb{P}\Big\{\alpha\big([0,1]^p\big)\geq t\Big\} \tag{3.3.1}$$
$$= -\frac{d(p-1)}{2}\Big(\frac{2p-d(p-1)}{2p}\Big)^{\frac{2p-d(p-1)}{d(p-1)}}\rho^{-\frac{2p}{d(p-1)}}.$$

where ρ is defined by (3.2.1).

Remark 3.3.3 . *Let $\kappa(d,p)>0$ be the best constant of the Gagliardo-Nirenberg inequality (Section C, Appendix)*

$$||f||_{2p}\leq C||\nabla f||_2^{\frac{d(p-1)}{2p}}||f||_2^{1-\frac{d(p-1)}{2p}} \qquad f\in W^{1,2}(\mathbb{R}^d) \tag{3.3.2}$$

where

$$W^{1,2}(\mathbb{R}^d)=\big\{f\in\mathcal{L}^2(\mathbb{R}^d);\quad \nabla f\in\mathcal{L}^2(\mathbb{R}^d)\big\}.$$

According to Theorem C.2 in Appendix,

$$\rho=\Big(\frac{2p-d(p-1)}{2p}\Big)^{\frac{2p-d(p-1)}{2p}}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p}}\kappa(d,p)^2. \tag{3.3.3}$$

Consequently, (3.3.1) can be rewritten as

$$\lim_{t\to\infty} t^{-\frac{2}{d(p-1)}}\log\mathbb{P}\Big\{\alpha\big([0,1]^p\big)\geq t\Big\}=-\frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}}. \tag{3.3.4}$$

The relation (3.3.3) will serve as a bridge between two very important and very different approaches used in this book: The large deviations by high moment asymptotics and the large deviations by Feynman-Kac formula (the later will be introduced in Chapter 4).

The problem on the exact value of Gagliardo-Nirenberg constant has some significance in physics but remains largely open. The case $d=1$ is an exception in which

$$\kappa(1,p)=p^{-1/2}\Big(\frac{\sqrt{2}}{(p-1)(p+1)}B\Big(\frac{1}{p-1},\frac{1}{2}\Big)\Big)^{-\frac{p-1}{2p}}$$

(Theorem C.4, Appendix), where $B(\cdot,\cdot)$ is beta function. By (2.2.16),

$$\alpha\big([0,1]^p\big)=\int_{\mathbb{R}^d}\Big[\prod_{j=1}^p L_j(1,x)\Big]dx$$

when $d = 1$, where $L_j(t,x)$ is the local time of $W_j(t)$ $(1 \le j \le p)$. The above leads to the following corollary.

Corollary 3.3.4 *When $d = 1$, for any $p \ge 2$,*

$$\lim_{t\to\infty} t^{-\frac{2}{p-1}} \log \mathbb{P}\Big\{ \int_{\mathbb{R}^d} \Big[\prod_{j=1}^p L_j(1,x)\Big] dx \ge t\Big\}$$
$$= -\frac{p}{4(p-1)}\Big(\frac{p+1}{2}\Big)^{\frac{3-p}{p-1}} B\Big(\frac{1}{p-1}, \frac{1}{2}\Big)^2.$$

Remark 3.3.5 . *The large deviation stated in (3.3.4) implies that*

$$\mathbb{E}\exp\Big\{\theta\Big(\alpha\big([0,1]^p\big)\Big)^{\frac{2}{d(p-1)}}\Big\}\begin{cases} < \infty & \theta < \frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}} \\ = \infty & \theta > \frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}}. \end{cases}$$

which appears to be improved version of Theorem 2.2.9.

Proof of Theorem 3.3.2. By Theorem 1.2.7, we need only to show that

$$\lim_{m\to\infty} \frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big] \tag{3.3.5}$$
$$= p\log\rho + \frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}.$$

Let $\tau_1,\cdots,\tau_p$ be independent exponential times with parameter 1. We assume the independence between $\{\tau_1,\cdots,\tau_p)$ and $(W_1,\cdots,W_p)$. In view of (2.2.19), Stirling formula and Theorem 3.2.2 lead to the requested upper bound

$$\limsup_{m\to\infty} \frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big] \tag{3.3.6}$$
$$\le p\log\rho + \frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}.$$

We now establish the lower bound for (3.3.5). By the moment inequality given in (2.2.11)

$$\begin{aligned}
&\mathbb{E}\Big[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]\\
&=\int_0^\infty\cdots\int_0^\infty dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\mathbb{E}\Big[\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big]\\
&\le\int_0^\infty\cdots\int_0^\infty dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\prod_{j=1}^p\Big(\mathbb{E}\Big[\alpha\big([0,t_j]^p\big)^m\Big]\Big)^{1/p}\\
&=\Big\{\int_0^\infty e^{-t}\Big(\mathbb{E}\Big[\alpha\big([0,t]^p\big)^m\Big]\Big)^{1/p}dt\Big\}^p\\
&=\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\Big\{\int_0^\infty t^{\frac{2p-d(p-1)}{2p}m}e^{-t}dt\Big\}^p\\
&=\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\Big[\Gamma\Big(\frac{2p-d(p-1)}{2p}m+1\Big)\Big]^p
\end{aligned}$$

where the fourth step follows from scaling property given in Proposition 2.2.6. Consequently,

$$\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\ge\Big[\Gamma\Big(\frac{2p-d(p-1)}{2p}m+1\Big)\Big]^{-p}\mathbb{E}\Big[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big].$$

By Stirling formula and by Theorem 3.2.2,

$$\begin{aligned}
&\liminf_{m\to\infty}\frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\\
&\ge p\log\rho+\frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}
\end{aligned}\tag{3.3.7}$$

Finally, (3.3.5) follows from (3.3.6) and (3.3.7). □

A direct application of the above result is the law of the iterated logarithm. To establish it, we recall that for $\bar{y}=(y_1,\cdots,y_p)$, $\mathbb{P}_{\bar{y}}$ is the probability distribution of the random path $\overline{W}(t)=\big(W_1(t),\cdots,W_p(t)\big)$ started at $W_1(0)=y_1,\cdots,W_p(0)=y_p$.

Lemma 3.3.6 *Under $p(d-2)<d$, for any $C>0$*

$$\lim_{t\to\infty}t^{\frac{2}{d(p-1)}}\log\inf_{|\bar{y}|\le C}\mathbb{P}_{\bar{y}}\Big\{\alpha\big([0,1]^p,\vec{x}\big)\ge t\Big\}=-\frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}}.$$

Proof. By (2.2.23) and (3.3.5),

$$\lim_{m\to\infty}\frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\sup_{\bar y}\mathbb{E}_{\bar y}\Big[\alpha\big([0,1]^p\big)^m\Big]$$
$$= p\log\rho + \frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}.$$

If we can prove that

$$\liminf_{m\to\infty}\frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\sup_{|\bar y|\le C}\mathbb{E}_{\bar y}\Big[\alpha\big([0,1]^p\big)^m\Big] \tag{3.3.8}$$
$$\ge p\log\rho + \frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}$$

by Theorem 1.2.7 and by Remark 3.3.3 we will have Lemma 3.3.6.

By (2.2.22),

$$\mathbb{E}_{\bar y}\Big[\big([0,1]^p\big)^m\Big]$$
$$=\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\prod_{j=1}^p\sum_{\sigma\in\Sigma_m}\int_{[0,1]^m_<}ds_1\cdots ds_m$$
$$\times\, p_{s_1}(x_{\sigma(1)}-y_j)\prod_{k=2}^m p_{s_k-s_{k-1}}(x_{\sigma(k)}-x_{\sigma(k-1)}).$$

Let the small number $\delta>0$ be fixed for a while. A straight forward verification gives that for $0<\epsilon<\delta/2$, there is $c>0$ such that

$$\inf_{|y|\le C}p_s(x-y)\ge cp_{s-\epsilon}(x)\qquad \forall s\ge\delta,\quad x\in\mathbb{R}^d.$$

Thus

$$\int_{[0,1]^m_<}ds_1\cdots ds_m p_{s_1}(x_{\sigma(1)}-y_j)\prod_{k=2}^m p_{s_k-s_{k-1}}(x_{\sigma(k)}-x_{\sigma(k-1)})$$
$$\ge c\int_{[\delta,1]^m_<}ds_1\cdots ds_m p_{s_1-\epsilon}(x_{\sigma(1)})\prod_{k=2}^m p_{s_k-s_{k-1}}(x_{\sigma(k)}-x_{\sigma(k-1)})$$
$$= c\int_{[\delta-\epsilon,1-\epsilon]^m_<}ds_1\cdots ds_m p_{s_1}(x_{\sigma(1)})\prod_{k=2}^m p_{s_k-s_{k-1}}(x_{\sigma(k)}-x_{\sigma(k-1)})$$

where the last step follows from time translation.

Summarizing our argument, we have obtained that

$$\inf_{|\bar y|\le C}\mathbb{E}_{\bar y}\Big[\alpha\big([0,1]^p\big)^m\Big]\ge c^p\mathbb{E}\Big[\alpha\big([\delta-\epsilon,1-\epsilon]^p\big)^m\Big] \tag{3.3.9}$$
$$\ge c^p\mathbb{E}\Big[\alpha\big([\delta,1-\delta]^p\big)^m\Big].$$

By triangular inequality,

$$\Big\{\mathbb{E}\Big[\alpha\big([0,1-\delta]^p\big)^m\Big]\Big\}^{1/m}$$
$$\le \Big\{\mathbb{E}\Big[\alpha\big([\delta,1-\delta]^p\big)^m\Big]\Big\}^{1/m} + \Big\{\mathbb{E}\Big[\alpha\big([0,1-\delta]^p \setminus [\delta,1-\delta]^p\big)\Big]^m\Big\}^{1/m}.$$

Partition the set $[0,1-\delta]^p \setminus [\delta,1-\delta]^p$ into p essentially disjoint boxes $B_1,\cdots,B_p$. For $1\le j\le p$, $B_j = A_1\times\cdots\times A_p$ with $A_l = [\delta,1-\delta]$ or $[0,1-\delta]$ for $l\ne j$ and $A_j=[0,\delta]$. By (2.2.11)

$$\mathbb{E}\Big[\alpha(B_j)^m\Big] \le \Big(\mathbb{E}\Big[\alpha\big([0,\delta]^p\big)^m\Big]\Big)^{1/p}\Big(\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\Big)^{\frac{p-1}{p}}$$
$$= \delta^{\frac{2p-p(d-1)}{2p}m}\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]$$

where the second step follows from the scaling property stated in Proposition 2.2.6. By Triangular inequality again,

$$\Big\{\mathbb{E}\Big[\big([0,1-\delta]^p\setminus[\delta,1-\delta]^p\big)\Big]^m\Big\}^{1/m} \le p\delta^{\frac{2p-p(d-1)}{2p}}\Big\{\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\Big\}^{1/m}.$$

In addition, by Proposition 2.2.6

$$\Big\{\mathbb{E}\Big[\big([0,1-\delta]^p\big)^m\Big]\Big\}^{1/m} = (1-\delta)^{\frac{2p-d(p-1)}{2}}\Big\{\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\Big\}^{1/m}.$$

Summarizing our argument since (3.3.9) and by (3.3.5), we have obtained that

$$\liminf_{m\to\infty}\frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\sup_{|\bar{y}|\le C}\mathbb{E}_{\bar{y}}\Big[\alpha\big([0,1]^p\big)^m\Big]$$
$$\ge \log\Big[(1-\delta)^{\frac{2p-d(p-1)}{2}} - \delta^{\frac{2p-p(d-1)}{2p}}\Big]$$
$$+\,p\log\rho + \frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}.$$

Letting $\delta\to 0^+$ on the right hand side gives (3.3.8). □

Theorem 3.3.7 *Under $p(d-2)<d$,*

$$\limsup_{t\to\infty} t^{-\frac{2p-d(p-1)}{2}}(\log\log t)^{-\frac{d(p-1)}{2}}\alpha\big([0,t]^p\big) = \Big(\frac{2}{p}\Big)^{-\frac{d(p-1)}{2}}\kappa(d,p)^{2p} \qquad a.s.$$

Proof. By the scaling property described in Proposition 2.2.6,

$$\begin{aligned}&\mathbb{P}\Big\{\alpha\big([0,t]^p\big) \geq \lambda t^{\frac{2p-d(p-1)}{2}}(\log\log t)^{\frac{d(p-1)}{2}}\Big\}\\ &= \mathbb{P}\Big\{\alpha\big([0,1]^p\big) \geq \lambda(\log\log t)^{\frac{d(p-1)}{2}}\Big\}.\end{aligned}$$

Replacing t by $\lambda(\log\log t)^{\frac{d(p-1)}{2}}$ in (3.3.4), we have that for any $\lambda > 0$

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\log\log t}\log\mathbb{P}\Big\{\alpha\big([0,t]^p\big) \geq \lambda t^{\frac{2p-d(p-1)}{2}}(\log\log t)^{\frac{d(p-1)}{2}}\Big\}\\ &= -\lambda^{\frac{2}{d(p-1)}}\frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}}.\end{aligned}$$

Notice that

$$\lambda_0 = \Big(\frac{2}{p}\Big)^{-\frac{d(p-1)}{2}}\kappa(d,p)^{2p}$$

is the solution of the equation

$$\lambda^{\frac{2}{d(p-1)}}\frac{d(p-1)}{2}\frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}} = 1$$

For any $\lambda_1 > \lambda_0$, there is a $\epsilon > 0$ such that

$$\begin{aligned}&\mathbb{P}\Big\{\alpha\big([0,t]^p\big) \geq \lambda_1 t^{\frac{2p-d(p-1)}{2}}(\log\log t)^{\frac{d(p-1)}{2}}\Big\}\\ &\leq \exp\{-(1+\epsilon)\log\log t\} = (\log t)^{-(1+\epsilon)}\\ &\mathbb{P}\Big\{\alpha\big([0,t]^p\big) \geq \lambda_1 t^{\frac{2p-d(p-1)}{2}}(\log\log t)^{\frac{d(p-1)}{2}}\Big\}\\ &\geq \exp\{-(1-\epsilon)\log\log t\} = (\log t)^{-(1-\epsilon)}\end{aligned}$$

as t is sufficiently large. Let $\theta > 1$ be fixed but arbitrary and define $t_k = \theta^k$ $k = 1, 2, \cdots$. We have

$$\sum_{k=1}^{\infty}\mathbb{P}\Big\{\alpha\big([0,t_k]^p\big) \geq \lambda_1 t_k^{\frac{2p-d(p-1)}{2}}(\log\log t_k)^{\frac{d(p-1)}{2}}\Big\} < \infty.$$

By Borel-Cantelli lemma,

$$\limsup_{k\to\infty} t_k^{-\frac{2p-d(p-1)}{2}}(\log\log t_k)^{-\frac{d(p-1)}{2}}\alpha\big([0,t_k]^p\big) \leq \lambda_1 \qquad a.s.$$

For any large t there is a $k \geq 1$ such that $t_k \leq t \leq t_k$. So

$$\begin{aligned}&t^{-\frac{2p-d(p-1)}{2}}(\log\log t)^{-\frac{d(p-1)}{2}}\alpha\big([0,t]^p\big)\\ &\leq t_k^{-\frac{2p-d(p-1)}{2}}(\log\log t_k)^{-\frac{d(p-1)}{2}}\alpha\big([0,t_{k+1}]^p\big)\\ &\leq \Big(\theta + o(1)\Big)^{\frac{2p-d(p-1)}{2}} t_{k+1}^{-\frac{2p-d(p-1)}{2}}(\log\log t_{k+1})^{-\frac{d(p-1)}{2}}\alpha\big([0,t_{k+1}]^p\big)\end{aligned}$$

as $k \to \infty$. Consequently,

$$\limsup_{t\to\infty} t^{-\frac{2p-d(p-1)}{2}} (\log\log t)^{-\frac{d(p-1)}{2}} \alpha\big([0,t]^p\big) \le \theta\lambda_1 \qquad a.s.$$

Letting $\theta \to 1^+$ and $\lambda_2 \to \lambda_0^+$ on the right hand side gives the desired upper bound

$$\limsup_{t\to\infty} t^{-\frac{2p-d(p-1)}{2}} (\log\log t)^{-\frac{d(p-1)}{2}} \alpha\big([0,t]^p\big) \le \lambda_0 \qquad a.s. \tag{3.3.10}$$

An obstacle for the lower bound is that the needed part of classic Borel-Cantelli lemma requires the independence of the event sequence, which does not hold in our case. To fix this problem we introduce an *extended Borel-Cantelli lemma* .

Lemma 3.3.8 *Let $\{A_k\}_{k\ge 1}$ be a sequence of events that is adaptive to the filtration $\{\mathcal{A}_k\}_{k\ge 1}$. Then almost surely,*

$$\{A_k \ \textit{occur infinitely often}\} = \Big\{\sum_{k=1}^\infty \mathbb{P}(A_{k+1}|\mathcal{A}_k) = \infty\Big\}.$$

This lemma was established in the book by Breiman (Corollary 5.29, p.96, [16]), where the proof appears as an elegant application of the martingale convergence theorem.

We now return to our work on the lower bound. Take $s_k = k^k$ ($k = 1, 2, \cdots$). By Lemma 3.3.6 and (2.2.21),

$$\lim_{t\to\infty} \frac{1}{\log\log t} \log \inf_{|\bar{x}|\le\sqrt{t}} \mathbb{P}_{\bar{x}}\Big\{\alpha\big([0,t]^p, \vec{x}\big) \ge \lambda t^{\frac{2p-d(p-1)}{2}} (\log\log t)^{\frac{d(p-1)}{2}}\Big\}$$
$$= -\lambda^{\frac{2}{d(p-1)}} \frac{p}{2} \kappa(d,p)^{-\frac{4p}{d(p-1)}}.$$

Consequently, for any $\lambda_2 < \lambda_o$, the series

$$\sum_{k=1}^\infty \inf_{|\bar{x}|\le\sqrt{s_{k+1}}} \mathbb{P}_{\bar{x}}\Big\{\alpha\big([0, s_{k+1} - s_k]^p, \vec{x}\big) \ge \lambda_2 s_{k+1}^{\frac{2p-d(p-1)}{2}} (\log\log s_{k+1})^{\frac{d(p-1)}{2}}\Big\}$$

diverges.

On the other hand, recall the classic law of the iterated logarithm for Brownian motions which states that

$$\limsup_{t\to\infty} \frac{|\overline{W}(t)|}{\sqrt{2t\log\log t}} = 1 \qquad a.s.$$

Consequently,

$$\mathbb{P}\Big\{|\overline{W}(s_k)| \le u\sqrt{\frac{s_{k+1}}{\log\log s_{k+1}}}\ \text{ eventually}\Big\} = 1.$$

Hence, with probability 1,

$$\sum_{k=1}^{\infty}\mathbb{P}_{\overline{W}(s_k)}\Big\{\alpha\big([0, s_{k+1}-s_k]^p\big) \ge \lambda_2 s_{k+1}^{\frac{2p-d(p-1)}{2}}(\log\log s_{k+1})^{\frac{d(p-1)}{2}}\Big\} = \infty.$$

By Markov property of Brownian motions, this is equivalent to

$$\sum_{k=1}^{\infty}\mathbb{P}\Big\{\alpha\big([s_{k+1}, s_k]^p\big) \ge \lambda_2 s_{k+1}^{\frac{2p-d(p-1)}{2}}(\log\log s_{k+1})^{\frac{d(p-1)}{2}}\Big|\mathcal{A}_k\Big\} = \infty \qquad a.s.$$

where $\mathcal{A}_k$ is the σ-field generated by $\{\overline{W}(s);\quad 0 \le s \le s_k\}$. By Lemma 3.3.8,

$$\limsup_{k\to\infty} s_{k+1}^{-\frac{2p-d(p-1)}{2}}(\log\log s_{k+1})^{-\frac{d(p-1)}{2}}\alpha\big([s_k, s_{k+1}]^p\big) \ge \lambda_2 \qquad a.s.$$

This leads to

$$\limsup_{t\to\infty} t^{-\frac{2p-d(p-1)}{2}}(\log\log t)^{-\frac{d(p-1)}{2}}\alpha\big([0,t]^p\big) \ge \lambda_2 \qquad a.s.$$

Letting $\lambda_2 \to \lambda_0^-$ on the right hand side gives the lower bound

$$\limsup_{t\to\infty} t^{-\frac{2p-d(p-1)}{2}}(\log\log t)^{-\frac{d(p-1)}{2}}\alpha\big([0,t]^p\big) \ge \lambda_0 \qquad a.s. \tag{3.3.11}$$

Finally, Theorem 3.3.7 follows from (3.3.10) and (3.3.11). □

3.4 Notes and comments

Section 3.1

The study of the high moment asymptotics was initiated by König and Mörters [107]. Their investigation leads to the large deviation for Brownian intersection local time limited to a finite domain and killed upon exit from the domain. Their approach relies on some sophisticated combinatorial estimate (given in II.2, [47]) for the tail of the empirical measure

$$L_n = \frac{1}{n}\sum_{k=1}^{n}\delta_{(Y_k, Y_{k+1})}$$

where $\{Y_k\}$ is an i.i.d. sequence taking values in a finite catalog. The treatment given here appeared in Chen [28] and in Chen and Mörters [33].

At its beginning stage, the study of high moment asymptotics has shown its power of solving hard problems in large deviations which are out of reach of the conventional methods. We shall see later in the proof of Theorem 7.5.3 how this powerful tool is applied to the super-critical dimensions. In addition, we point out the references Chen [26], Chen [28], Chen [29], Bass, Chen and Rosen [8] for the key role played by the method of high moment asymptotics in the recent progress on the large deviations for the local times self-intersection local times and Riesz potentials of Additive Processes, where the asymptotics of the form

$$\int_{(\mathbb{R}^d)^m} \pi(d\lambda_1)\cdots\pi(d\lambda_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m Q\Big(\sum_{j=1}^k \lambda_{\sigma(j)}\Big)\Big]^p$$

is investigated. For further application, a challenge we face is how to extend the upper bound established in Theorem 3.1.2 to the settings of reasonable generality where, for example, the state space E can be non-compact or non-discrete.

Section 3.2

Concerning developing the general theory of high moment asymptotics, main result Theorem 3.2.2 serves as a good example supporting generalization of Theorem 3.1.2. The folding technique leading to (3.2.3) is partially inspired by a similar idea going back to Donsker and Varadhan [54].

We propose a "cheaper version" of Theorem 3.2.2 in the following exercise.

Exercise 1. Assume that $d(p-2) < d$. Write

$$H_m = \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\Big]^p \qquad m = 1, 2, \cdots.$$

By (2.2.20) for each integer $m \geq 1$,

$$H_m \leq (m!)^p\Big[\int_{\mathbb{R}^d} G^p(x)dx\Big]^m$$

which gives the bound

$$\limsup_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\mathbb{E}\Big[\alpha([0,\tau_1]\times\cdots\times[0,\tau_p])^m\Big] \leq \log\int_{\mathbb{R}^d} G^p(x)dx.$$

In the following we establish the existence of the limit.

(1). Prove for integers $m, n \geq 1$,

$$\frac{1}{((m+n)!)^p} H_{m+n} \leq \frac{1}{(m!)^p} H_m \frac{1}{(n!)^p} H_n.$$

Hint: Let Σ'_{m+n} be the set of 1-1 maps from $\{1, \cdots, m+n\}$ to $\{1, \cdots, m+n\}$ that map $\{1, \cdots m\}$ to $\{1, \cdots m\}$. First show that

$$H_{m+n} = \binom{m+n}{m} \int_{(\mathbb{R}^d)^{m+n}} dx_1 \cdots dx_{m+n} \Big[\sum_{\sigma \in \Sigma'_{m+n}} \prod_{k=1}^{m+n} G(x_{\pi(k)} - x_{\pi(k-1)}) \Big]$$
$$\times \Big[\sum_{\sigma \in \Sigma_m} \prod_{k=1}^{m+n} G(x_{\sigma(k)} - x_{\sigma(k-1)}) \Big]^{p-1}.$$

Then use Hölder's inequality.

(2). Prove the weaker version of Theorem 3.2.2: There is a constant $-\infty < \mathcal{C} < \infty$ such that

$$\lim_{m \to \infty} \frac{1}{m} \log \frac{1}{(m!)^p} \mathbb{E}\Big[\alpha([0, \tau_1] \times \cdots \times [0, \tau_p])^m\Big] = \mathcal{C}.$$

Hint: Use Lemma 1.3.1.

Section 3.3

Theorem 3.3.2 was obtained in [24] by a very different method. Prior to that, Le Gall ([117], [123]), Brydges and Slade ([17]) had estimated the moment of $\alpha\big([0,1]^p\big)$. Together with Chebyshev inequality, some of their results give bound to the tail probability of $\alpha\big([0,1]^p\big)$ that are sharp up to constant.

In addition to their pioneering paper on the high moment asymptotics, König and Mörters ([108]) applied their method to the exponential moments for a functional of intersection local times of independent Broanian motions.

The Gagliardo-Nirenberg inequality has a close tie to Sobolev inequality (see Section C). The full class of Gagliardo-Nirenberg inequalities defined in the analysis literature is wider than the one given in (3.3.2). We point out the books by Ziemer ([169]) and by Saloff-Coste ([146]) for the general reference. Finding the best constants for Gagliardo-Nirenberg inequality appears to be a difficult problem when $d \geq 2$ and still remains open in general. It has been attracting considerable attention partially due to its connection to some problem in physics. The best constant for Nash's inequality, which is a special case of Gagliardo-Nirenberg inequality, was found by Carlen and

Loss ([21]). See also papers by Cordero-Erausquin ([36]) and by Del Pino and Dolbeault ([42]) for a recent progress on the best constants for a class of Gagliardo-Nirenberg inequalities and logarithmic Sobolev inequalities. For a connection between the best constants of Gagliardo-Nirenberg inequalities and logarithmic Sobolev inequalities, the interested reader is referred to Del Pino and Dolbeault [43]. In the critical case $p(d-2) = d$, (3.3.2) becomes the Sobolev inequality

$$||f||_{\frac{2d}{d-2}} \leq C||\nabla f||_2.$$

The best constant in this case has been known (see, e.g., Theorem 1.3.2, p. 20, [146]).

Unfortunately, none of these progresses was made for the kind of Gagliardo-Nirenberg inequalities given in (3.3.2) as $p(d-2) < d$. .

The paper directly related to $\kappa(d,p)$ for $p(d-2) < d$ are [125] and [157]. In [125], Levine obtained a sharp estimate for $\kappa(2,3)$, which claims that

$$\sqrt{\frac{1}{4.6016}} < \kappa(2,3) < \sqrt{\frac{1}{4.5981}}.$$

Based on that, he conjecture that $\kappa(2,3) = \pi^{-4/9}$.

Weinstein ([157]) studied the problem of the best constants for the Gagliardo-Nirenberg inequalities given in (3.3.2). It was shown (Theorem B, [157]) that under $p(d-2) < d$, the best constant in (3.3.2) is attained at an infinitely smooth, positive and spherically symmetric function f_0, which solves the non-linear equation

$$\frac{d(p-1)}{2}\Delta f - \frac{2p-d(p-1)}{2} f + f^{2p-1} = 0. \tag{3.4.1}$$

In addition, f_0 has the smallest L^2-norm among all solutions of the above equation (such solution is called a *ground state solution*). Further

$$\kappa(d,p) = \left(p||f_0||_{2p}^{-2(p-1)}\right)^{\frac{1}{2p}}. \tag{3.4.2}$$

Using this result, Weinstein obtained the following numerical approximation in the case $d = p = 2$,

$$\kappa(2,2) = \sqrt[4]{\frac{1}{\pi \times 1.86225...}}. \tag{3.4.3}$$

Chapter 4

Large deviations: self intersection

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According to the discussion in Section 2.3 and Section 2.4, self-intersection local time $\beta([0,t]^p_<)$ exists in the case $d=1$ (Proposition 2.3.5) and is renormalizable in the case $d=2$ (Theorem2.4.1). The gaols of this chapter are to establish the large deviation principles for the self-intersection local times in 1-dimension and for the renormalized self-intersection local time in 2-dimension, and to apply them to the laws of the iterated logarithm. Our approach consists of the following three main ingredients:

1. Feynman-Kac formula combined with some some ideas developed from the area of Probability in Banach spaces. The method is used for the the p-square integral of the local time given in (2.3.5).

2. Comparison between $\gamma\big([0,t]^2_<\big)$ and $\alpha\big([0,1]^2\big)$. The connections between the self-intersection local times and mutual intersection local times have been seen in the discussion in Section 2.3 and Section 2.4. We shall explore this link in a much more quantitative way.

3. Argument by sub-additivity. This method has been developed into effective tool in large deviation theory. Even so one might still be surprised to see throughout this book how often that the relations such as (2.4.11) holds in our models. Naturally, the sub-additivity method becomes an important part of the methodology adopted in this book.

4.1 Feynman-Kac formula

Let $W(t)$ be a d-dimensional Brownian motions and let $f(x)$ be a bounded continuous function. Write

$$T_t g(x) = \mathbb{E}\bigg(\exp\Big\{ \int_0^t f\big(x + W(s)\big) ds \Big\} g\big(x + W(t)\big) \bigg) \qquad t \ge 0. \tag{4.1.1}$$

We now show that for each $t \ge 0$, T_t defines a continuous linear operator on the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$. The linearity is obvious. One can see that when $t = 0$, $T_0 = I$ is an identical operator. In particular, T_0 is an continuous operator on $\mathcal{L}^2(\mathbb{R}^d)$. Assume $t > 0$.

$$\begin{aligned}
\int_{\mathbb{R}^d} \big[T_t g(x)\big]^2 dx &= \int_{\mathbb{R}^d} \bigg[\mathbb{E}\bigg(\exp\Big\{ \int_0^t f\big(x + W(s)\big) ds \Big\} g\big(x + W(t)\big) \bigg) \bigg]^2 dx \\
&\le \int_{\mathbb{R}^d} \mathbb{E}\bigg(\exp\Big\{ 2\int_0^t f\big(x + W(s)\big) ds \Big\} g^2\big(x + W(t)\big) \bigg) dx \\
&= \mathbb{E} \int_{\mathbb{R}^d} \exp\Big\{ 2\int_0^t f\big(x + W(s)\big) ds \Big\} g^2\big(x + W(t)\big) dx \\
&= \int_{\mathbb{R}^d} \exp\Big\{ 2\int_0^t f(x) ds \Big\} g^2(x) dx \le e^{tC} \int_{\mathbb{R}^d} g^2(x) dx
\end{aligned}$$

where the second step follows from Cauchy-Schwartz inequality, the fourth step from translation invariance, and the last step from the boundedness of f.

Therefore, $\{T_t; t \ge 0\}$ is a set of linear bounded operators on the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$. For any $t > 0$, $g, h \in \mathcal{L}^2(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} T_t g(x) \cdot h(x) dx \\
&= \mathbb{E}\bigg(\int_{\mathbb{R}^d} \exp\Big\{ \int_0^t f\big(x + W(s)\big) ds \Big\} g\big(x + W(t)\big) h(x) dx \bigg) \\
&= \mathbb{E}\bigg(\int_{\mathbb{R}^d} \exp\Big\{ \int_0^t f\big(x + W(s) - W(t)\big) ds \Big\} g(x) h\big(x - W(t)\big) dx \bigg\} \\
&= \mathbb{E}\bigg(\int_{\mathbb{R}^d} \exp\Big\{ \int_0^t f\big(x + W(t-s) - W(t)\big) ds \Big\} g(x) h\big(x - W(t)\big) dx \bigg\} \\
&= \int_{\mathbb{R}^d} g(x) \cdot T_t h(x) dx
\end{aligned}$$

where the last step follows from the identity in law

$$\Big\{W(t-s)-W(t);\quad 0\le s\le t\Big\} \stackrel{d}{=} \Big\{W(s);\quad 0\le s\le t\Big\}.$$

The above relation means that T_t is symmetric. By the boundedness of T_t established earlier, T_t is a self-adjoint operator on $\mathcal{L}^2(\mathbb{R}^d)$ (see the discussion in Section B).

Lemma 4.1.1 *$\{T_t; t\ge 0\}$ is a semi-group in the sense that for any $s,t\ge 0$*

$$T_{s+t} = T_s\circ T_t.$$

Proof. For any $g\in\mathcal{L}^(\mathbb{R}^d)$,

$$\begin{aligned}
T_{s+t}g(x) &= \mathbb{E}\left(\exp\left\{\int_0^{s+t} f\big(x+W(u)\big)du\right\}g\big(x+W(s+t)\big)\right)\\
&= \mathbb{E}\left(\exp\left\{\int_0^{s} f\big(x+W(u)\big)du\right\}\right.\\
&\times\exp\left\{\int_s^{s+t} f\big(x+W(s)+W(s+u)-W(s)\big)du\right\}\\
&\left.\times\, g\big(x+W(s)+W(s+t)-W(s)\big)\right)\\
&= \mathbb{E}\left(\exp\left\{\int_0^{s} f\big(x+W(u)\big)du\right\}(T_tg)\big(x+W(s)\big)\right) = T_s(T_tg)(x)
\end{aligned}$$

where the third step follows from independence of increment and Fubini theorem. □

A self-adjoint operator T is said to be *non-negative* if $\langle g, Tg\rangle \ge 0$ for any g in the domain $\mathcal{D}(T)$ of T (In the case of bounded operator, of course, $\mathcal{D}(T)=\mathcal{L}^2(\mathbb{R}^d)$). By semi-group property and self-adjointness,

$$\langle g, T_tg\rangle = \langle g,\ T_{t/2}\circ T_{t/2}g\rangle = ||T_{t/2}g||_2^2\ge 0 \qquad g\in\mathcal{L}^2(\mathbb{R}^d).$$

Therefore, for each $t\ge 0$, T_t is non-negative.

An important aspect of a semi-group is its infinitesimal generator A, which is a linear operator satisfying

$$\lim_{t\to 0^+}\frac{T_tg-g}{t} = Ag$$

on a subspace of $\mathcal{L}^2(\mathbb{R}^d)$, where the limit is taken in an appropriate sense. To make the matter less complicated, we first consider the case when $g(x)$ is a rapidly decreasing function. A infinitely smooth function $g(x)$ is said to be *rapidly decreasing*, if $g(x)$ and its derivatives of all orders decay at infinity faster than any negative power of $|x|$.

We now state the famous *Feynman-Kac formula*.

Theorem 4.1.2 *For any rapidly decreasing function $g(x)$ on $\mathbb{R}^d$,*

$$\lim_{t\to 0^+}\frac{T_tg(x)-g(x)}{t}=\frac{1}{2}\Delta g(x)+f(x)g(x)\qquad x\in\mathbb{R}^d. \tag{4.1.2}$$

Proof. First we consider the special case when $f\equiv 0$, in which

$$T_t^0g(x)\equiv T_tg(x)=\int_{\mathbb{R}^d}p_t(y)g(x+y)dy=\int_{\mathbb{R}^d}p_t(y-x)g(y)dy$$

By Fourier transform,

$$\widehat{T_t^0g}(\lambda)\equiv\int_{\mathbb{R}^d}e^{i\lambda\cdot x}T_tg(x)dx=\widehat{g}(\lambda)\exp\Big\{-\frac{t|\lambda|^2}{2}\Big\}\qquad\lambda\in\mathbb{R}^d$$

where

$$\widehat{g}(\lambda)=\int_{\mathbb{R}^d}e^{i\lambda\cdot x}g(x)dx.$$

By Fourier inversion (Theorem B.1, Appendix),

$$T_t^0g(x)-g(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-i\lambda\cdot x}\widehat{g}(\lambda)\Big[\exp\Big\{-\frac{t|\lambda|^2}{2}\Big\}-1\Big]d\lambda.$$

Consequently, by dominated convergence, and by the fact that the Fourier transform of a rapidly decreasing function is rapidly decreasing,

$$\lim_{t\to 0^+}\frac{T_t^0g(x)-g(x)}{t}=-\frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-i\lambda\cdot x}|\lambda|^2\widehat{g}(\lambda)d\lambda=\frac{1}{2}\Delta g(x) \tag{4.1.3}$$

where the last step follows from (B.8), Appendix. So we have (4.1.2) when $f\equiv 0$.

We now prove (4.1.2) with full generality. First notice that

$$\begin{aligned}&\exp\Big\{\int_0^t f\big(x+W(s)\big)ds\Big\}\\&=1+\int_0^t f\big(x+W(s)\big)\exp\Big\{\int_s^t f\big(x+W(u)\big)du\Big\}ds.\end{aligned}$$

Multiplying both side by $g\big(x+W(t)\big)$, taking expectation and applying increment independence on the right hand side,

$$T_t g(x) = T_t^0 g(x) + \int_0^t \mathbb{E}\Big\{f\big(x+W(s)\big)T_{t-s}g\big(x+W(s)\big)\Big\}ds.$$

Therefore, (4.1.2) follows from (4.1.3) and the fact that

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^t \mathbb{E}\Big\{f\big(x+W(s)\big)T_{t-s}g\big(x+W(s)\big)\Big\}ds = f(x)g(x).$$

□

Let $\mathcal{S}(\mathbb{R}^d)$ be the set of all rapidly decreasing functions on $\mathbb{R}^d$, which is also known as *Schwartz space.* Write

$$Ag(x) = \frac{1}{2}\Delta g(x) + f(x)g(x) \qquad g\in\mathcal{S}(\mathbb{R}^d). \tag{4.1.4}$$

The linear operator A is called the *infinitesimal generator* of the semi-group $\{T_t,\ t\ge 0\}$. In (4.1.4) we momentarily limit the domain of A to $\mathcal{S}(\mathbb{R}^d)$. By integration by parts one can see that

$$\langle g, Ah\rangle = \langle Ag, h\rangle \qquad g,h\in\mathcal{S}(\mathbb{R}^d). \tag{4.1.5}$$

This means that A is a symmetric operator (see Section E, Appendix for definition of symmetric operator on the general Hilbert space).

As pointed out in Section E of Appendix, an unbounded symmetric operator does not have to be self-adjoint. It appears to be extremely difficult to examine the self-adjointness of the unbounded operator A. On the other hand, the self-adjointness is crucially needed for the later development. Our treatment is to extend A into a self-adjoint operator. Notice that for each $g\in\mathcal{S}(\mathbb{R}^d)$,

$$\langle Ag, g\rangle = \int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx \le C||g||_2^2$$

where $C = \sup_{x\in\mathbb{R}^d}|f(x)|$. A symmetric operator with such property is said to be *upper semi-bounded* . By Friedrichs extension theorem (Theorem 2, Section 7, Chapter XI, [168]), an upper semi-bounded symmetric operator admits a self-adjoint extension known as *Friedrichs' extension.* We still use the same notation for the Friedrichs' extension of A and still call it infinitesimal generator of the semi-group T_t. We denote $\mathcal{D}(A)$ for the domain of the self-adjoint operator A. Clearly, $\mathcal{S}(\mathbb{R}^d)\subset\mathcal{D}(A)\subset\mathcal{L}^2(\mathbb{R}^d)$.

Recall our notation

$$W^{1,2}(\mathbb{R}^d) = \Big\{g \in \mathcal{L}^2(\mathbb{R}^d);\quad \nabla g \in \mathcal{L}^2(\mathbb{R}^d)\Big\}$$

and set

$$\mathcal{F}_d = \Big\{g \in \mathcal{L}^2(\mathbb{R}^d);\quad ||g||_2 = 1 \ \text{ and } \ ||\nabla g||_2 < \infty\Big\}.$$

In the following lemma we determine some basic structure of the operator A.

Lemma 4.1.3 *We have that $\mathcal{D}(A) \subset W^{1,2}(\mathbb{R}^d)$, and that*

$$\langle g, Ag\rangle = \int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \qquad g \in \mathcal{D}(A), \tag{4.1.6}$$

$$\sup_{\substack{g\in\mathcal{D}(A)\\ ||g||_2=1}} \langle g, Ag\rangle = \sup_{g\in\mathcal{F}_d}\Big\{\int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx\Big\}. \tag{4.1.7}$$

Proof. Let $g \in \mathcal{D}(A)$ and write, for each $\epsilon > 0$,

$$g_\epsilon(x) = \int_{\mathbb{R}^d} g(y)p_\epsilon(y-x)dx$$

where $p_\epsilon(x)$ is the density of $W(\epsilon)$. It is easy to see that $g_\epsilon \in \mathcal{S}(\mathbb{R}^d)$. By (4.1.4),

$$\langle Ag_\epsilon, g\rangle = \int_{\mathbb{R}^d} f(x)g_\epsilon(x)g(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} \Delta g_\epsilon(x)g(x)dx.$$

We shall let $\epsilon \to 0^+$ on the both sides. For the left-hand side, observe that $g_\epsilon \to g$ in $\mathcal{L}^2(\mathbb{R}^d)$. Consequently,

$$\langle Ag_\epsilon, g\rangle = \langle g_\epsilon, Ag\rangle \longrightarrow \langle g, Ag\rangle \qquad (\epsilon \to 0^+).$$

By the boundedness of $f(x)$, the first term on the right-hand side approaches to

$$\int_{\mathbb{R}^d} f(x)g^2(x)dx$$

To treat the second integral on the right-hand side, we apply the tool Fourier transformation. By Parseval identity ((B.9), Appendix), the convolution identity ((B.11), Appendix), and the fact ((B.8), Appendix) that $\widetilde{\Delta g_\epsilon} = |\lambda|^2\widetilde{g}_\epsilon$,

$$\int_{\mathbb{R}^d} \Delta g_\epsilon(x)g(x)dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\lambda|^2 \exp\Big\{-\frac{\epsilon}{2}|\lambda|^2\Big\}|\widehat{g}(\lambda)|^2 d\lambda.$$

Therefore, we must have

$$\int_{\mathbb{R}^d} |\lambda|^2 |\widehat{g}(\lambda)|^2 d\lambda < \infty$$

By Theorem B.2, Appendix, $g \in W^{1,2}(\mathbb{R}^d)$ and

$$\begin{aligned}\langle g, Ag\rangle &= \int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\lambda|^2 |\widehat{g}(\lambda)|^2 d\lambda \\ &= \int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx.\end{aligned}$$

Finally, (4.1.7) follows from the relation $\mathcal{S}(\mathbb{R}^d) \subset D(A) \subset W^{1,2}(\mathbb{R}^d)$, the fact that $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{1,2}(\mathbb{R}^d)$ under the Sobolev norm $\sqrt{||g||_2^2 + ||\nabla g||_2^2}$, and the continuity (in g) of the quadratic form on the right-hand side of (4.1.6) under the Sobolev norm. □

The quadratic form $\langle g, Ag\rangle$ given in (4.1.6) is often known as *Dirichlet form* of the semi-group $\{T_t\}$.

By the fact that $||T_t|| \le e^{Ct}$ $(t \ge 0)$ for some constant $C > 0$, the following linear operator

$$G_\lambda g(x) = \int_0^\infty e^{-\lambda t} T_t g(x) dt \qquad g \in \mathcal{L}^2(\mathbb{R}^d) \tag{4.1.8}$$

is well defined and is a bounded self-adjoint operator on $\mathcal{L}^2(\mathbb{R}^d)$.

Lemma 4.1.4 *For any $g \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ $T_t g(x)$ is differentiable on $(0, \infty)$ as a function of t and*

$$\frac{d}{dt}\Big(T_t g(x)\Big) = (A \circ T_t)g(x) = (T_t \circ A)g(x). \tag{4.1.9}$$

In addition, there is a $C > 0$ such that for any $\lambda > C$,

$$(\lambda I - A) \circ G_\lambda g = G_\lambda \circ (\lambda I - A)g = g \qquad g \in \mathcal{S}(\mathbb{R}^d). \tag{4.1.10}$$

Proof. We first pointed out the following analytic lemma (the proof can be find in p.239, Section 3, Chapter IX, [168]): Let $u(t)$ be a continuous function on $(0, \infty)$ such that its right derivative

$$D^+ u(t) \equiv \lim_{\Delta t \to 0^+} \frac{u(t + \Delta t) - u(t)}{\Delta t}$$

is finite and continuous on $(0,\infty)$. Then $u(t)$ is differentiable on $(0,\infty)$.

To establish (4.1.9), therefore, it suffices to show that

$$\lim_{\Delta t\to 0^+}\frac{T_{t+\Delta t}g(x)-T_tg(x)}{\Delta t}(A\circ T_t)g(x)=(T_t\circ A)g(x).$$

Indeed, this follows from the equality

$$\frac{T_{t+\Delta t}g(x)-T_tg(x)}{\Delta t}=T_t\circ\frac{T_{\Delta t}-I}{\Delta t}g(x)=\frac{T_{\Delta t}-I}{\Delta t}\circ T_tg(x)$$

and (4.1.2) in Theorem 4.1.2.

By (4.1.9),

$$\begin{aligned}(\lambda I-A)\circ G_\lambda g(x)&=G_\lambda g(x)-\int_0^\infty e^{-\lambda t}(A\circ T_t)g(x)dt\\ &=G_\lambda g(x)-\int_0^\infty e^{-\lambda t}\frac{d}{dt}\Big(T_tg(x)\Big)dt.\end{aligned}$$

Thus, (4.1.10) follows from integration by parts. □

An very important fact is that the semi-group $\{T_t;\ t\ge 0\}$ can be written in terms of some functions of A. Our discussion here is based on the material introduced in Section E, Appendix. Generally speaking, a function $\xi(A)$ of the self-adjoint operator is another self-adjoint operator. Being precise, we consider the spectral integral representation (Theorem E.1, Appendix)

$$A=\int_{-\infty}^\infty\lambda E(d\lambda) \tag{4.1.11}$$

of the self-adjoint operator A, where $\{E(\lambda);\quad -\infty<\lambda<\infty\}$ is a resolution of the identity , a family of projection operators on $\mathcal{L}^2(\mathbb{R}^d)$ with distribution-function-type properties. In addition,

$$\mathcal{D}(A)=\left\{g\in\mathcal{L}^2(\mathbb{R}^d);\quad\int_{-\infty}^\infty|\lambda|^2\mu_g(d\lambda)<\infty\right\} \tag{4.1.12}$$

where μ_g is a finite measure (known as spectral measure) on $\mathbb{R}$ induced by the distribution function $F(\lambda)\equiv\langle g,E(\lambda)g\rangle$ with

$$\mu_g(\mathbb{R})=||g||_2^2\qquad g\in\mathcal{L}^2(\mathbb{R}^d). \tag{4.1.13}$$

Given a Borel function $\xi(\lambda)$ on $\mathbb{R}$, the function $\xi(A)$ of A is a self-adjoint operator defined by the spectral integral

$$\xi(A)=\int_{-\infty}^\infty\xi(\lambda)E(d\lambda) \tag{4.1.14}$$

with the domain $\mathcal{D}\big(\xi(A)\big)$ given by

$$\mathcal{D}\big(\xi(A)\big) = \left\{ g \in \mathcal{L}^2(\mathbb{R}^d); \quad \int_{-\infty}^{\infty} |\xi(\lambda)|^2 \mu_g(d\lambda) < \infty \right\}. \tag{4.1.15}$$

Given $\lambda > 0$, for example, we take $\xi(\gamma) = (\lambda - \gamma)^{-1}$ and define the following operator

$$(\lambda I - A)^{-1} = \int_{-\infty}^{\infty} \frac{1}{\lambda - \gamma} E(d\gamma) \tag{4.1.16}$$

Here we use the fact ((4.1.7)) that

$$\lambda_0 \equiv \sup_{\substack{g \in \mathcal{D}(A) \\ ||g||_2 = 1}} \langle g, Ag \rangle < \infty. \tag{4.1.17}$$

By Theorem E.2, Appendix, μ_g is supported by $(-\infty, \lambda_0]$ for every $g \in \mathcal{L}^2(\mathbb{R}^d)$. Consequently, by examining the integrability listed in (4.1.15) the operator $(\lambda I - A)^{-1}$ is defined on the whole space $\mathcal{L}^2(\mathbb{R}^d)$ as $\lambda > \lambda_0$. Further, by (E.14) in Appendix,

$$\left|\left|(\lambda I - A)^{-1} g\right|\right|_2^2 = \int_{-\infty}^{\infty} \frac{1}{(\lambda - \gamma)^2} \mu_g(d\lambda) \le \frac{1}{(\lambda - \lambda_0)^2} ||g||_2^2 \qquad g \in \mathcal{L}^2(\mathbb{R}^d).$$

This implies that $(\lambda I - A)^{-1}$ is a bounded operator. As its notation suggested, $(\lambda I - A)^{-1}$ is the inverse operator of $\lambda I - A$ in the sense that $(\lambda I - A)^{-1} \circ (\lambda I - A)$ and $(\lambda I - A) \circ (\lambda I - A)^{-1}$ are identical operators on $\mathcal{D}(A)$ and on $\mathcal{L}^2(\mathbb{R}^d)$, respectively. To see this, first notice that by (4.1.11)

$$(\lambda I - A) = \int_{-\infty}^{\infty} (\lambda - \gamma) E(d\gamma).$$

Our claim follows immediately from Theorem E.3, Appendix.

Further, applying $(\lambda I - A)^{-1}$ to the both sides of (4.1.10) we see that $(\lambda I - A)^{-1}$ and G_λ agree on $\mathcal{S}(\mathbb{R}^d)$ when $\lambda > 0$ is sufficiently large. By the fact that $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{L}^2(\mathbb{R}^d)$, and by the continuity of the operators $(\lambda I - A)^{-1}$ and G_λ, we have that for large $\lambda > 0$,

$$(\lambda I - A)^{-1} = G_\lambda. \tag{4.1.18}$$

For given $t \ge 0$, we now take $\xi(\lambda) = \exp\{t\lambda\}$ and consider the self-adjoint operator $\exp\{tA\}$ defined by

$$\exp\{tA\} = \int_{-\infty}^{\infty} \exp\{t\lambda\} E(d\lambda). \tag{4.1.19}$$

By Theorem E.2 the support of the spectral measure μ_g is bounded from above by λ_0 given in (4.1.17) for all $g \in \mathcal{L}^2(\mathbb{R}^d)$. Consequently, $\exp\{tA\}$ is defined on whole space $\mathcal{L}^2(\mathbb{R}^d)$. Moreover, by (E.14) in Appendix,

$$\Big|\Big|\exp\{tA\}g\Big|\Big|^2 = \int_{-\infty}^{\infty} \exp\{2t\lambda\}\mu_g(d\lambda) \le \exp\{2t\lambda_0\}||g||_2^2.$$

Hence, $\exp\{tA\}$ is a bounded operator. Further, by (E.15) in Appendix

$$\Big\langle g, \exp\{tA\}g\Big\rangle = \int_{-\infty}^{\infty} \exp\{t\lambda\}\mu_g(d\lambda) \ge 0 \qquad g \in \mathcal{L}^2(\mathbb{R}^d).$$

Consequently, $\exp\{tA\}$ is non-negative.

Now, we state the following representation theorem.

Theorem 4.1.5 *For each $t \ge 0$, $T_t = e^{tA}$.*

Proof. First, we reduce Theorem 4.1.5 to the equality

$$\Big\langle g,\ \exp\{tA\}g\Big\rangle = \langle g, T_t g\rangle \qquad g \in \mathcal{L}^2(\mathbb{R}^d). \tag{4.1.20}$$

Indeed, for any $g, h \in \mathcal{L}^2(\mathbb{R}^d)$,

$$\Big\langle h,\ \exp\{tA\}g\Big\rangle = \frac{1}{4}\Big\{\Big\langle g+h,\ \exp\{tA\}(g+h)\Big\rangle + \Big\langle g-h,\ \exp\{tA\}(g-h)\Big\rangle\Big\},$$

$$\langle h, T_t g\rangle = \frac{1}{4}\Big\{\langle g+h,\ T_t(g+h)\rangle + \langle g-h,\ T_t(g-h)\rangle\Big\}.$$

If (4.1.20) holds,

$$\Big\langle h,\ \exp\{tA\}g\Big\rangle = \langle h, T_t g\rangle.$$

Since g and h are arbitrary, this leads to the conclusion of Theorem 4.1.5.

To establish (4.1.20), we begin with the operator equation (4.1.18) (We always keep $\lambda > 0$ sufficiently large in our argument), which can be written as

$$\Big(I - \frac{A}{\lambda}\Big)^{-1} = \lambda\int_0^{\infty} e^{-\lambda s}T_s ds.$$

Iterating n times (in the sense of operator composition) on the both sides,

$$\begin{aligned}
&\Big(I - \frac{A}{\lambda}\Big)^{-n} \\
&= \lambda^n \int_{(\mathbb{R}^+)^n} \exp\{-\lambda(s_1 + \cdots + s_n)\}T_{s_1}\circ\cdots\circ T_{s_n} ds_1\cdots ds_n \\
&= \lambda^n \int_{(\mathbb{R}^+)^n} \exp\{-\lambda(s_1 + \cdots + s_n)\}T_{s_1+\cdots+s_n} ds_1\cdots ds_n.
\end{aligned}$$

By the spectral integral representation (4.1.16) and by Theorem E.3, Appendix, the left-hand side can be represented as

$$\int_{-\infty}^{\infty}\Big(1-\frac{\gamma}{\lambda}\Big)^{-n}E(d\gamma).$$

By (E.15) in Appendix, therefore,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\Big(1-\frac{\gamma}{\lambda}\Big)^{-n}\mu_g(d\gamma)=\Big\langle g,\ \Big(I-\frac{A}{\lambda}\Big)^{-n}g\Big\rangle\\
&=\lambda^n\int_{(\mathbb{R}^+)^n}\exp\{-\lambda(s_1+\cdots+s_n)\}\big\langle g,\ T_{s_1+\cdots+s_n}g\big\rangle ds_1\cdots ds_n\\
&=\int_{(\mathbb{R}^+)^n}\exp\{-(s_1+\cdots+s_n)\}\big\langle g,\ T_{\lambda^{-1}(s_1+\cdots+s_n)}g\big\rangle ds_1\cdots ds_n.
\end{aligned}$$

Replacing λ by n/t,

$$\int_{-\infty}^{\infty}\Big(1-\frac{t\gamma}{n}\Big)^{-n}\mu_g(d\gamma)=\mathbb{E}\langle g,T_{t\zeta_n}g\rangle \tag{4.1.21}$$

where the expectation on the right hand side is taken with respect to the random variable

$$\zeta_n=\frac{\tau_1+\cdots+\tau_n}{n}$$

which appears as the sample average of i.i.d. exponential random variables $\tau_1,\cdots,\tau_n$ with parameter 1.

By the classic law of large numbers, $\zeta_n\xrightarrow{p}1$ as $n\to\infty$. By the continuity of $\langle g,T_sg\rangle$ as a function of s and by the bound $\langle g,T_sg\rangle\le C_1e^{C_2s}$, one has

$$\lim_{n\to\infty}\mathbb{E}\langle g,T_{t\zeta_n}g\rangle=\langle g,T_tg\rangle.$$

By the fact that the supporting set of the measure μ_g is bounded from above, and by the dominated convergence theorem,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\Big(1-\frac{t\gamma}{n}\Big)^{-n}\mu_g(d\gamma)=\int_{-\infty}^{\infty}e^{t\gamma}\mu_g(d\gamma)=\Big\langle g,\ \exp\{tA\}g\Big\rangle$$

where the second equality follows from (E.15), Appendix.

Finally, letting $n\to\infty$ on the both sides of (4.1.21) leads to (4.1.20). $\square$

An interesting application of Feynman-Kac formula is the following large deviation result.

Theorem 4.1.6 *For any bounded, continuous function f on $\mathbb{R}^d$*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\Big\{\int_0^t f\big(W(s)\big)ds\Big\}$$
$$=\sup_{g\in\mathcal{F}_d}\Big\{\int_{\mathbb{R}^d} f(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big\},$$

where

$$\mathcal{F}_d=\Big\{g\in\mathcal{L}^2(\mathbb{R}^d);\ \ \int_{\mathbb{R}^d} g^2(x)dx\ \textit{and}\ \int_{\mathbb{R}^d}|\nabla g(x)|^2dx<\infty\Big\}. \tag{4.1.22}$$

Proof. By the boundedness of f there is $C>0$ such that

$$C^{-1}\le\exp\Big\{\int_0^1 f\big(W(s)\big)ds\Big\}\le C. \tag{4.1.23}$$

Let $g\in\mathcal{S}(\mathbb{R}^d)$ be compactly supported and satisfy

$$\int_{\mathbb{R}^d} g^2(x)dx=1.$$

We have

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\int_1^t f\big(W(s)\big)ds\Big\}\\
&\ge||g||_\infty^{-2}\mathbb{E}\bigg(g\big(W(1)\big)\exp\Big\{\int_1^t f\big(W(s)\big)ds\Big\}g\big(W(t)\big)\bigg)\\
&=||g||_\infty^{-2}\mathbb{E}\bigg(g\big(W(1)\big)\exp\Big\{\int_1^{t-1} f\big(W(1)+W(1+s)-W(1)\big)ds\Big\}\\
&\qquad\qquad\times g\big(W(1)+W(t)-W(1)\big)\bigg)\\
&=||g||_\infty^{-2}\mathbb{E}\Big(g\big(W(1)\big)T_{t-1}g\big(W(1)\big)\Big)\\
&=||g||_\infty^{-2}\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-|x|^2/2}g(x)T_{t-1}g(x)dx.
\end{aligned}$$

On the supporting set of g, $e^{-|x|^2/2}$ is bounded from below by a positive number. Therefore, we have established

$$\mathbb{E}\exp\Big\{\int_0^t f\big(W(s)\big)ds\Big\}\ge\delta\int_{\mathbb{R}^d} g(x)T_{t-1}g(x)dx=\delta\langle g,T_{t-1}g\rangle. \tag{4.1.24}$$

By Theorem 4.1.5 and by (E.15) in Appendix (with $\xi(\lambda) = \exp\{t\lambda\}$), for each $t > 0$

$$\langle g, T_t g\rangle = \int_{-\infty}^{\infty} e^{t\lambda}\mu_g(d\lambda) \tag{4.1.25}$$
$$\geq \exp\Big\{t\int_{-\infty}^{\infty}\lambda\mu_g(d\lambda)\Big\} = \exp\Big\{t\langle g, Ag\rangle\Big\}$$
$$= \exp\Big\{t\Big(\int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big)\Big\}$$

where the inequality follows from Jensen inequality and the fact ((4.1.13)) that $\mu_g(\mathbb{R}^+) = 1$, which shows that the spectral measure μ_g is a probability measure; and where the third step follows from (E.15) in Appendix (with $\xi(\lambda) = \lambda$).

By (4.1.24), therefore,

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\Big\{\int_0^t f\big(W(s)\big)ds\Big\}$$
$$\geq \int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx.$$

Taking supremum over g gives the desired lower bound:

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\Big\{\int_0^t f\big(W(s)\big)ds\Big\} \tag{4.1.26}$$
$$\geq \sup_{g\in\mathcal{F}_d}\Big\{\int_{\mathbb{R}^d} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big\}.$$

We now establish the upper bound. For each $t > 0$

$$\mathbb{E}\exp\Big\{\int_1^t f\big(W(s)\big)ds\Big\}$$
$$\leq \mathbb{E}\bigg(1_{\{|W(1)|\leq t^2\}}\exp\Big\{\int_1^t f\big(W(s)\big)ds\Big\}1_{\{|W(t)|\leq t^2\}}\bigg)$$
$$+ \mathbb{E}\bigg(1_{\{|W(1)|>t^2\}}\exp\Big\{\int_1^t f\big(W(s)\big)ds\Big\}\bigg)$$
$$+ \mathbb{E}\bigg(\exp\Big\{\int_1^t f\big(W(s)\big)ds\Big\}1_{\{|W(t)|>t^2\}}\bigg).$$

By boundedness of f, there is $C > 0$ such that

$$\mathbb{E}\bigg(1_{\{|W(1)|>t^2\}}\exp\Big\{\int_1^t f\big(W(s)\big)ds\Big\}\bigg) \leq e^{Ct}\mathbb{P}\{|W(1)| > t^2\} \leq Ce^{Ct}e^{-t^4/2}$$

where the second estimate follows from estimation of Gaussian tail. Similarly,

$$\mathbb{E}\bigg(\exp\bigg\{\int_1^t f\big(W(s)\big)ds\bigg\}1_{\{|W(t)|>t^2\}}\bigg)\le e^{Ct}\mathbb{P}\{|W(t)|>t^2\}\le Ce^{Ct}e^{-t^3/2}.$$

Thus,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\int_1^t f\big(W(s)\big)ds\bigg\} \tag{4.1.27}$$
$$\le\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\bigg(1_{\{|W(1)|\le t^2\}}\exp\bigg\{\int_1^t f\big(W(s)\big)ds\bigg\}1_{\{|W(t)|\le t^2\}}\bigg).$$

For each $t>0$, let $g_t\in\mathcal{S}(\mathbb{R}^d)$ such that $g_t(x)=1$ for $|x|\le t^2$, $g_t(x)=0$ for $|x|>2t^2$ and $0\le g_t(x)\le 1$ for all x.

$$\mathbb{E}\bigg(1_{\{|W(1)|\le t^2\}}\exp\bigg\{\int_1^t f\big(W(s)\big)ds\bigg\}1_{\{|W(t)|\le t^2\}}\bigg) \tag{4.1.28}$$
$$\le ||g_t||_2^2\mathbb{E}\bigg(\bar{g}_t\big(W(1)\big)\exp\bigg\{\int_1^t f\big(W(s)\big)ds\bigg\}\bar{g}_t\big(W(t)\big)\bigg)$$
$$=\frac{1}{(2\pi)^{d/2}}||g_t||_2^2\int_{\mathbb{R}^d}\exp\bigg\{-\frac{|x|^2}{2}\bigg\}\bar{g}_t(x)(T_{t-1}\bar{g}_t)(x)dx$$
$$\le\frac{1}{(2\pi)^{d/2}}||g_t||_2^2\langle\bar{g}_t,T_{t-1}\bar{g}_t\rangle$$

where $\bar{g}_t(x)=g_t(x)/||g_t||$.

Notice that $||g||_2^2=O(t^{2d})$ as $t\to\infty$. By Theorem 4.1.5 and (E.15) in Appendix,

$$\langle\bar{g}_t,T_{t-1}\bar{g}_t\rangle=\int_{-\infty}^{\infty}e^{t\lambda}\mu_{\bar{g}_t}(d\lambda).$$

By (4.1.13) $\mu_{\bar{g}_t}$ is a probability measure. In view of (4.1.17) and (4.1.7), by Theorem E.2 in Appendix the smallest supporting set of $\mu_{\bar{g}_t}$ is bounded from above by

$$\sup_{\substack{g\in\mathcal{D}(A)\\ ||g||_2=1}}\langle g,Ag\rangle=\sup_{g\in\mathcal{F}_d}\bigg\{\int_{\mathbb{R}^d}f(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\bigg\}$$

Hence,

$$\langle\bar{g}_t,T_{t-1}\bar{g}_t\rangle \tag{4.1.29}$$
$$\le\exp\bigg(t\sup_{g\in\mathcal{F}_d}\bigg\{\int_{\mathbb{R}^d}f(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\bigg\}\bigg).$$

Combining (4.1.23), (4.1.27), (4.1.28) and (4.1.29) we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\int_0^t f\big(W(s)\big)ds\right\} \tag{4.1.30}$$
$$\le \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d} f(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.$$

Finally, the desired conclusion follows from (4.1.26) and (4.1.30). □

4.2 One dimensional case.

The local time $L(t,x)$ of an 1-dimensional Brownian motion $W(t)$ exists and for any integer $p\ge 2$. By Proposition 2.3.5, the self-intersection local time $\beta\big([0,t]^p_<\big)$ is equal to

$$\frac{1}{p!}\int_{-\infty}^{\infty}L^p(t,x)dx.$$

By (2.3.8) in Proposition 2.3.3,

$$\int_{-\infty}^{\infty}L^p(t,x)dx\overset{d}{=}t^{\frac{p+1}{2}}\int_{-\infty}^{\infty}L^p(1,x)dx. \tag{4.2.1}$$

Thus, we focus on the large deviation principle for the L_p-norm of the Brownian local times up to the time 1. The following theorem is the main result of this section.

Theorem 4.2.1 *When $d=1$, for any real number $p>1$,*

$$\lim_{a\to\infty}a^{-\frac{2}{p-1}}\log\mathbb{P}\Big\{\int_{-\infty}^{\infty}L^p(1,x)dx\ge a\Big\}$$
$$=-\frac{1}{4(p-1)}\Big(\frac{p+1}{2}\Big)^{\frac{3-p}{p-1}}B\Big(\frac{1}{p-1},\frac{1}{2}\Big)$$

where $B(\cdot,\cdot)$ is beta function.

Proof. By Corollary 1.2.4, Theorem 4.2.1 holds if we can prove for each $\theta>0$ that

$$\lim_{a\to\infty}a^{-\frac{2}{p-1}}\log\mathbb{E}\exp\left\{a^{\frac{p+1}{p(p-1)}}\theta\Big(\int_{-\infty}^{\infty}L^p(1,x)dx\Big)^{1/p}\right\}$$
$$=\Big(\frac{\theta}{p}\Big)^{\frac{2p}{p+1}}\left(\frac{\sqrt{2}}{(p-1)(p+1)}B\Big(\frac{1}{p-1},\frac{1}{2}\Big)\right)^{-\frac{2(p-1)}{p+1}}.$$

By (4.2.1) and (C.8) in Theorem C.4, Appendix, this is equivalent to

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\left(\int_{-\infty}^{\infty}L^p(t,x)dx\right)^{1/p}\right\} \\ &=\sup_{g\in\mathcal{F}}\left\{\left(\int_{-\infty}^{\infty}|g(x)|^{2p}dx\right)^{1/p}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.\end{aligned}\tag{4.2.2}$$

where

$$\mathcal{F}=\left\{g;\quad\int_{-\infty}^{\infty}|g(x)|^2dx=1\ \text{and}\ \int_{-\infty}^{\infty}|g'(x)|^2dx<\infty\right\}$$

The proof of (4.2.2) is a combination of Feynman-Kac formula and some method developed along the probability in Banach space. Let $q>1$ be the conjugate number and f be a bounded continuous function on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty}|f(x)|^qdx=1.\tag{4.2.3}$$

By Hölder inequality,

$$\left(\int_{-\infty}^{\infty}L^p(t,x)dx\right)^{1/p}\ge\int_{-\infty}^{\infty}f(x)L(t,x)dx=\int_0^t f\big(W(s)\big)ds.$$

By Theorem 4.1.6, therefore,

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\left(\int_{-\infty}^{\infty}L^p(t,x)dx\right)^{1/p}\right\} \\ &\ge\sup_{g\in\mathcal{F}}\left\{\int_{-\infty}^{\infty}f(x)g^2(x)dx-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.\end{aligned}$$

Notice that the functions f satisfying our conditions are dense in the unit sphere of $\mathcal{L}^q(\mathbb{R})$. Taking supremum over f on the right hand side gives

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\left(\int_{-\infty}^{\infty}L^p(t,x)dx\right)^{1/p}\right\} \\ &\ge\sup_{g\in\mathcal{F}}\left\{\left(\int_{-\infty}^{\infty}|g(x)|^{2p}dx\right)^{1/p}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.\end{aligned}\tag{4.2.4}$$

The upper bound is more delicate. The self-intersection outside any compact set of $\mathbb{R}$ is not negligible. Indeed, for any $M>0$, by the argument for the lower bound (4.2.4),

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\left(\int_M^\infty L^p(t,x)dx\right)^{1/p}\right\}$$
$$\geq \sup_{g\in\mathcal{F}}\left\{\left(\int_M^\infty |g(x)|^{2p}dx\right)^{1/p}-\frac{1}{2}\int_{-\infty}^\infty |g'(x)|^2dx\right\}.$$

By shifting the space variable x, one can see that the variation on the right high side is equal to the right hand side of (4.2.4). Consequently, $L(t,x)$ is not exponentially tight when embedded into $\mathcal{L}^p(\mathbb{R})$.

To resolve this problem, we wrap the real line $\mathbb{R}$ into a big but compact circle. Let $M>0$ be large but fixed. We have

$$\int_{-\infty}^\infty L^p(t,x)dx=\sum_{k\in\mathbb{Z}}\int_0^M L^p(t,x+kM)dx\leq\int_0^M \tilde{L}^p(t,x)dx \tag{4.2.5}$$

where

$$\tilde{L}(t,x)=\sum_{k\in\mathbb{Z}}L(t,x+kM).$$

For any $s>0$, the process

$$W_s(u)=W(s+u)-W(s)\qquad u\geq 0$$

is an 1-dimensional Brownian motion. If we write $L_s(t,x)$ for the local time of $W_s(u)$ and write

$$\tilde{L}_s(t,x)=\sum_{k\in\mathbb{Z}}L_s(t,kM+x)$$

then for any $t>0$,

$$\int_0^M\left[\tilde{L}(s+t,x)-\tilde{L}(s,x)\right]^pdx=\int_0^M\tilde{L}_s^p\big(t,x-W(t)\big)dx=\int_0^M\tilde{L}_s^p(t,x)dx$$

where the last step follows from periodicity of $\tilde{L}_s^p\big(t,\cdot\big)$. By independence and stationarity of the Brownian increments, therefore, the quantity

$$\int_0^M\left[\tilde{L}(s+t,x)-\tilde{L}(s,x)\right]^pdx$$

is independent of $\{W(u);\ \ 0\leq u\leq s\}$ and has the same distribution as

$$\int_0^M\tilde{L}^p(t,x)dx.$$

By triangular inequality, for any $s, t > 0$,

$$\Big(\int_0^M \tilde{L}^p(s+t,x)dx\Big)^{1/p} \tag{4.2.6}$$
$$\le \Big(\int_0^M \tilde{L}^p(s,x)dx\Big)^{1/p} + \Big(\int_0^M \big[\tilde{L}(s+t,x) - \tilde{L}(s,x)\big]^p dx\Big)^{1/p}.$$

Based all properties discussed above, we conclude that the non-negative process

$$\Big(\int_0^M \tilde{L}^p(t,x)dx\Big)^{1/p} \qquad t \ge 0$$

is sub-additive. By Theorem 1.3.5,

$$\mathbb{E}\exp\Big\{\theta\Big(\int_0^M \tilde{L}^p(1,x)dx\Big)^{1/p}\Big\} < \infty \tag{4.2.7}$$

$$\lim_{t\to\infty} \frac{1}{t}\log\mathbb{E}\exp\Big\{\theta\Big(\int_0^M \tilde{L}^p(t,x)dx\Big)^{1/p}\Big\} < \infty. \tag{4.2.8}$$

for every $\theta > 0$.

We view $\tilde{L}(t,\cdot)$ as a stochastic process taking values in the separable Banach space $\mathcal{L}^p[0,M]$. For any $x \in \mathbb{R}$ we adopt the notations $\mathbb{P}_x$ and $\mathbb{E}_X$ for the distribution and expectation, respectively, of the Brownian motion $W(t)$ starting at x (i. e., $W(0) = x$). We let $t = 1$ and claim that for any $\epsilon > 0$ there is a compact set $K \subset \mathcal{L}^p[0,M]$ such that

$$\inf_{x\in\mathbb{R}} \mathbb{P}_x\big\{\tilde{L}(1,\cdot) \in K\big\} \ge 1 - \epsilon. \tag{4.2.9}$$

Indeed, by continuity of $L(1,x)$ as a function of x and by the fact that only finitely many $L(t, \cdot + kM)$ is non-zero at any given moment, $\tilde{L}(1,x)$ is continuous. We have that for any $\gamma > 0$

$$\sup_{x\in\mathbb{R}} \mathbb{P}_x\Big\{\sup_{|y-z|\le\delta} |\tilde{L}(1,y) - \tilde{L}(1,z)| \ge \gamma\Big\}$$
$$= \sup_{x\in\mathbb{R}} \mathbb{P}\Big\{\sup_{|y-z|\le\delta} |\tilde{L}(1,x+y) - \tilde{L}(1,x+z)| \ge \gamma\Big\}$$
$$= \mathbb{P}\Big\{\sup_{|y-z|\le\delta} |\tilde{L}(1,y) - \tilde{L}(1,z)| \ge \gamma\Big\} \longrightarrow 0 \qquad (\delta \to 0^+).$$

By Arzelá-Ascoli theorem (Theorem D.2, Appendix), there is a compact set $\widetilde{K}$ compact in $C[0,M]$ such that such that

$$\inf_{x\in\mathbb{R}} \mathbb{P}_x\big\{\tilde{L}(1,\cdot) \in \widetilde{K}\big\} \ge 1 - \epsilon.$$

Notice that

$$\int_0^M \tilde{L}(1,x)dx = \int_{-\infty}^{\infty} L(1,x)dx = 1.$$

Hence, our claim follows from the fact that the set

$$K \equiv \widetilde{K} \cap \left\{ f \in C[0,M]; \quad f \ge 0 \text{ and } \int_0^M f(x)dx = 1 \right\}$$

is relatively compact in $\mathcal{L}^p[0,M]$.

Notice that for any $x \in \mathbb{R}$ and $\lambda > 0$,

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{ \lambda \bigg(\int_0^M \tilde{L}^p(1,y)dy \bigg)^{1/p} \right\} \\
&= \mathbb{E} \exp\left\{ \lambda \bigg(\int_0^M \tilde{L}^p(1,x+y)dy \bigg)^{1/p} \right\} \\
&= \mathbb{E} \exp\left\{ \lambda \bigg(\int_0^M \tilde{L}^p(1,y)dy \bigg)^{1/p} \right\}
\end{aligned}$$

where the last step follows from periodicity. By (4.2.7), the right hand side is finite. Thus,

$$\sup_{x\in\mathbb{R}} \mathbb{E}_x \exp\left\{ \lambda \bigg(\int_0^M \tilde{L}^p(1,y)dy \bigg)^{1/p} \right\} < \infty \qquad \forall \lambda > 0. \tag{4.2.10}$$

By Theorem 1.1.7, the uniform tightness of the family

$$\Big\{ \mathcal{L}_x\big(\tilde{L}(1,\cdot)\big); \quad x \in \mathbb{R} \Big\}$$

and (4.2.10) imply that there is a convexed, positively balanced and compact set $K \subset \mathcal{L}^p[0,M]$ such that

$$\sup_{x\in\mathbb{R}} \mathbb{E}_x \exp\Big\{ q_K\big(\tilde{L}(1,\cdot)\big) \Big\} < \infty. \tag{4.2.11}$$

For any $s,t > 0$ and $x \in \mathbb{R}$, by sub-additivity of Minkowski functional given in (1.1.8) and by Markov property

$$\begin{aligned}
&\mathbb{E}_x \exp\Big\{ q_K\big(\tilde{L}(s+t,\cdot)\big) \Big\} \\
&\le \mathbb{E}_x \exp\Big\{ q_K\big(\tilde{L}(s,\cdot)\big) + q_K\big(\tilde{L}(s+t,\cdot) - \tilde{L}(s,\cdot)\big) \Big\} \\
&= \mathbb{E}_x \bigg(\exp\Big\{ q_K\big(\tilde{L}(s,\cdot)\big) \Big\} \mathbb{E}_{W(s)} \exp\Big\{ q_K\big(\tilde{L}(t,\cdot)\big) \Big\} \bigg) \\
&\le \mathbb{E}_x \exp\Big\{ q_K\big(\tilde{L}(s,\cdot)\big) \Big\} \sup_{y\in\mathbb{R}} \mathbb{E}_y \exp\Big\{ q_K\big(\tilde{L}(t,\cdot)\big) \Big\}.
\end{aligned}$$

Thus,

$$\sup_{x\in\mathbb{R}} \mathbb{E}_x \exp\Big\{q_K\big(\tilde{L}(s+t,\cdot)\big)\Big\}$$
$$\le \sup_{x\in\mathbb{R}} \mathbb{E}_x \exp\Big\{q_K\big(\tilde{L}(s,\cdot)\big)\Big\} \sup_{x\in\mathbb{R}} \mathbb{E}_x \exp\Big\{q_K\big(\tilde{L}(t,\cdot)\big)\Big\}.$$

The sub-additivity established here implies (Lemma 1.3.1) that

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbb{R}} \mathbb{E}_x \exp\Big\{q_K\big(\tilde{L}(t,\cdot)\big)\Big\} = \inf_{s>0} \frac{1}{s} \log \sup_{x\in\mathbb{R}} \mathbb{E}_x \exp\Big\{q_K\big(\tilde{L}(s,\cdot)\big)\Big\}.$$

By (4.2.11), the right hand side is finite. Hence,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\Big\{q_K\big(\tilde{L}(t,\cdot)\big)\Big\} < \infty. \tag{4.2.12}$$

Let $\lambda > 0$ be arbitrary and write

$$\begin{aligned} &\mathbb{E} \exp\left\{\left(\int_0^M \tilde{L}^p(t,x)dx\right)^{1/p}\right\} \qquad (4.2.13)\\ &= \mathbb{E}\left[\exp\left\{\left(\int_0^M \tilde{L}^p(t,x)dx\right)^{1/p}\right\} 1_{\{\frac{1}{t}\tilde{L}(t,\cdot)\in\lambda K\}}\right] \\ &+ \mathbb{E}\left[\exp\left\{\left(\int_0^M \tilde{L}^p(t,x)dx\right)^{1/p}\right\} 1_{\{\frac{1}{t}\tilde{L}(t,\cdot)\notin\lambda K\}}\right]. \end{aligned}$$

For each $g \in \lambda K$, by Hahn-Banach theorem (Corollary 2, p.108, Section 6, Chapter IV, [168]) and by the facts that $\mathcal{L}^q[0,M]$ is the *topological dual space* of $\mathcal{L}^p[0,M]$ and that bounded and continuous functions are dense in $\mathcal{L}^q[0,M]$, there is a bounded and continuous function f such that

$$f(0) = f(M) \quad \text{and} \quad \int_0^M |f(x)|^q dx = 1 \tag{4.2.14}$$

and

$$\left(\int_0^M |g(x)|^p dx\right)^{1/p} < \epsilon + \int_0^M f(x)g(x)dx.$$

By the finite cover theorem, there are bounded and continuous functions $f_1, \cdots, f_m$ on $[0, M]$ satisfying (4.2.14) such that

$$\left(\int_0^M |g(x)|^p dx\right)^{1/p} < \epsilon + \max_{1\le i\le m} \int_0^M f_i(x)g(x)dx \qquad \forall g \in \lambda K.$$

Consequently,

$$\mathbb{E}\bigg[\exp\bigg\{\Big(\int_0^M \tilde{L}^p(t,x)dx\Big)^{1/p}\bigg\}1_{\{\frac{1}{t}\tilde{L}(t,\cdot)\in\lambda K\}}\bigg]$$
$$\le e^{\epsilon t}\sum_{i=1}^m \mathbb{E}\exp\bigg\{\int_0^M f_i(x)\tilde{L}(t,x)dx\bigg\}.$$

By the first equation in (4.2.14), f_i can be extended into a bounded, continuous and periodic function (with period M) on $\mathbb{R}$. By periodicity,

$$\int_0^M f_i(x)\tilde{L}(t,x)dx = \sum_{k\in\mathbb{Z}}\int_0^M f_i(x)L(t,kM+x)dx$$
$$= \int_{-\infty}^{\infty} f_i(x)L(t,x)dx = \int_0^t f_i\big(W(s)\big)ds.$$

By Theorem 4.1.6, therefore,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\int_0^M f_i(x)\tilde{L}(t,x)dx\bigg\}$$
$$= \sup_{g\in\mathcal{F}}\bigg\{\int_{-\infty}^{\infty} f_i(x)g^2(x)dx - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\bigg\}.$$

Using periodicity again,

$$\int_{-\infty}^{\infty} f_i(x)g^2(x)dx = \int_0^M f_i(x)\sum_{k\in\mathbb{Z}} g^2(kM+x)dx$$
$$\le \bigg(\int_0^M\Big(\sum_{k\in\mathbb{Z}} g^2(kM+x)\Big)^p dx\bigg)^{1/p}$$

where the last step follows from Hölder inequality. Summarizing our argument, we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\bigg[\exp\bigg\{\Big(\int_0^M \tilde{L}^p(t,x)dx\Big)^{1/p}\bigg\}1_{\{\frac{1}{t}\tilde{L}(t,\cdot)\in\lambda K\}}\bigg]$$
$$\le \epsilon + \sup_{g\in\mathcal{F}}\bigg\{\bigg(\int_0^M\Big(\sum_{k\in\mathbb{Z}} g^2(kM+x)\Big)^p dx\bigg)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\bigg\}.$$

Letting $\epsilon\to 0^+$ on the right hand side gives

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\bigg[\exp\bigg\{\Big(\int_0^M \tilde{L}^p(t,x)dx\Big)^{1/p}\bigg\}1_{\{\frac{1}{t}\tilde{L}(t,\cdot)\in\lambda K\}}\bigg] \tag{4.2.15}$$
$$\le \sup_{g\in\mathcal{F}}\bigg\{\bigg(\int_0^M\Big(\sum_{k\in\mathbb{Z}} g^2(kM+x)\Big)^p dx\bigg)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\bigg\}.$$

We now return to (4.2.13). By Cauchy-Schwartz inequality,

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg\{\Big(\int_0^M \tilde{L}^p(t,x)dx\Big)^{1/p}\bigg\}1_{\{\frac{1}{t}\tilde{L}(t,\cdot)\notin\lambda K\}}\bigg]\\
&\le\bigg[\mathbb{E}\exp\bigg\{2\Big(\int_0^M \tilde{L}^p(t,x)dx\Big)^{1/p}\bigg\}\bigg]^{1/2}\bigg[\mathbb{P}\Big\{\frac{1}{t}\tilde{L}(t,\cdot)\notin\lambda K\Big\}\bigg]^{1/2}.
\end{aligned}$$

Observe that

$$\mathbb{P}\Big\{\frac{1}{t}\tilde{L}(t,\cdot)\notin\lambda K\Big\}\le\mathbb{P}\Big\{q_K\big((\lambda t)^{-1}\tilde{L}(t,\cdot)\big)\ge 1\Big\}=\mathbb{P}\Big\{q_K\big(\tilde{L}(t,\cdot)\big)\ge\lambda t\Big\}$$

where the second step follows from the positive homogeneity of the Minkowski functional given in (1.1.8).

By (4.2.11) and Chebyshev inequality, therefore,

$$\lim_{\lambda\to\infty}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\frac{1}{t}\tilde{L}(t,\cdot)\notin\lambda K\Big\}=-\infty.$$

This, combined with (4.2.8), leads to

$$\lim_{\lambda\to\infty}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\bigg[\exp\bigg\{\Big(\int_0^M \tilde{L}^p(t,x)dx\Big)^{1/p}\bigg\}1_{\{\frac{1}{t}\tilde{L}(t,\cdot)\notin\lambda K\}}\bigg]=-\infty.$$

By (4.2.13) and (4.2.15), therefore,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\Big(\int_0^M \tilde{L}^p(t,x)dx\Big)^{1/p}\bigg\}\\
&\le\sup_{g\in\mathcal{F}}\bigg\{\Big(\int_0^M\Big(\sum_{k\in\mathbb{Z}}g^2(kM+x)\Big)^p dx\Big)^{1/p}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\bigg\}.
\end{aligned}$$

From (4.2.5) we have

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\Big(\int_{-\infty}^{\infty} L^p(t,x)dx\Big)^{1/p}\bigg\}\\
&\le\sup_{g\in\mathcal{F}}\bigg\{\Big(\int_0^M\Big(\sum_{k\in\mathbb{Z}}g^2(kM+x)\Big)^p dx\Big)^{1/p}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\bigg\}.
\end{aligned}$$

By Lemma 4.2.2 below, letting $M\to\infty$ on the right hand side gives the upper bound

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\Big(\int_{-\infty}^{\infty} L^p(t,x)dx\Big)^{1/p}\bigg\}\\
&\le\sup_{g\in\mathcal{F}}\bigg\{\Big(\int_{-\infty}^{\infty}|g(x)|^{2p}dx\Big)^{1/p}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\bigg\}.
\end{aligned}\tag{4.2.16}$$

Finally, (4.2.2) follows from (4.2.4) and (4.2.16). □

Lemma 4.2.2

$$\limsup_{M\to\infty}\sup_{g\in\mathcal{F}}\left\{\left(\int_0^M\Big(\sum_{k\in\mathbb{Z}}g^2(kM+x)\Big)^p dx\right)^{1/p}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}$$
$$\leq\sup_{g\in\mathcal{F}}\left\{\left(\int_{-\infty}^{\infty}|g(x)|^{2p}dx\right)^{1/p}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.$$

Proof. Without loss of generality we may assume that the right hand side, denoted by J, is finite. We only need to prove that given $\epsilon>0$, there is a $M>0$ such that for all $g\in\mathcal{F}$,

$$\left(\int_0^M\Big(\sum_{k\in\mathbb{Z}}g^2(kM+x)\Big)^p dx\right)^{1/p}-\frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2dx \tag{4.2.17}$$
$$\leq\epsilon+\Big(\frac{1}{1-\epsilon}\Big)^{\frac{p-1}{p+1}}J.$$

The hard part is that $M>0$ has to be independent of g. We shall determine the value of M later. Let

$$\bar{g}(x)=\left(\sum_{k\in\mathbb{Z}}g^2(kM+x)\right)^{1/2}.$$

Then $\bar{g}$ is absolutely continuous with

$$\int_0^M\bar{g}^2(x)dx=1\quad\text{and}\quad|\bar{g}'(x)|^2\leq\sum_{k\in\mathbb{Z}}|g'(kM+x)|^2 \tag{4.2.18}$$

where the second fact follows from direct computation of $\bar{g}'(x)$ and Cauchy-Schwartz inequality.

Next we need to construct a function $f\in\mathcal{F}$ which is equal to $\bar{g}$ on $[M^{1/2},M-M^{1/2}]$ and is negligible outside this interval. Write $E=[0,M]\cup[M-M^{1/2},M]$. Similar to (3.2.13), there is $a\in[0,M]$ (possibly depends on g) such that

$$\int_E|\bar{g}(x+a)|^2dx\leq 2M^{-1/2}. \tag{4.2.19}$$

We may assume that $a=0$, for otherwise we replace $\bar{g}(\cdot)$ by $\bar{g}(\cdot+a)$.

Define

$$C(x)=\begin{cases} xM^{-1/2} & 0\le x\le M^{1/2}\\ 1 & M^{1/2}\le x\le M-M^{1/2}\\ M^{1/2}-xM^{-1/2} & M-M^{1/2}\le x\le M\\ 0 & \text{otherwise.}\end{cases}$$

It is straightforward to verify that for each $x\in\mathbb{R}$,

$$0\le C(x)\le 1,\qquad |C'(x)|\le M^{-1/2},\qquad \big|\big(C^2(x)\big)'\big|\le 2M^{-1/2}.$$

Define $f\in\mathcal{F}$ by

$$f(x)=\Big(\int_{-\infty}^{\infty}C^2(y)\bar{g}^2(y)dy\Big)^{-1/2}C(x)\bar{g}(x)$$

and set

$$\alpha=\int_{-\infty}^{\infty}C^2(y)\bar{g}^2(y)dy.$$

We have

$$\begin{aligned}|f'(x)|^2&=\frac{1}{\alpha}\Big\{C^2(x)|\bar{g}'(x)|^2+|C'(x)|^2\bar{g}^2(x)+\frac{1}{2}\big(C^2(x)\big)'\big(\bar{g}^2(x)\big)'\Big\}\\ &=\frac{1}{\alpha}1_{[0,M]}(x)\Big\{|\bar{g}'(x)|^2+M^{-1}\bar{g}^2(x)+M^{-1/2}\big|\big(\bar{g}^2(x)\big)'\big|\Big\}.\end{aligned}$$

In addition,

$$\begin{aligned}&\int_0^M\big|\big(\bar{g}^2(x)\big)'\big|dx=2\int_0^M|\bar{g}(x)\bar{g}'(x)|dx\\ &\le 2\Big(\int_0^M|\bar{g}(x)|^2dx\Big)^{1/2}\Big(\int_0^M|\bar{g}'(x)|^2dx\Big)^{1/2}\\ &=2\Big(\int_0^M|\bar{g}'(x)|^2dx\Big)^{1/2}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_0^M\big|f'(x)|^2dx \qquad (4.2.20)\\ &\le\frac{1}{\alpha}\Big\{M^{-1}+\int_0^M|\bar{g}'(x)|^2dx+2M^{-1/2}\Big(\int_0^M|\bar{g}'(x)|^2dx\Big)^{1/2}\Big\}.\end{aligned}$$

On the other hand, notice that $f = \alpha^{-1/2}\bar{g}$ on $[-M, M] \setminus E$.

$$\Big(\int_0^M |\bar{g}(x)|^{2p} dx\Big)^{1/p} \le \Big(\alpha^p \int_{-\infty}^{\infty} |f(x)|^{2p} dx + \int_E |\bar{g}(x)|^{2p} dx\Big)^{1/p}$$
$$\le \alpha\Big(\int_{-\infty}^{\infty} |f(x)|^{2p} dx\Big)^{1/p} + \Big(\int_E |\bar{g}(x)|^{2p} dx\Big)^{1/p}.$$

In view of (4.2.19),

$$\Big(\int_E |\bar{g}(x)|^{2p} dx\Big)^{1/p} \le \big((2M)^{-1/2}\big)^{1/p} \sup_{0\le x\le M} |\bar{g}(x)|^{2/q}$$

(recall that $q > 1$ is the conjugate of p).

Observe that for any $x \in [0, M]$, if $x + 1 \in [0, M]$

$$|\bar{g}(x)| \le \int_x^{x+1} |\bar{g}(y)| dy + \int_x^{x+1} \Big(\int_x^y |\bar{g}'(z)| dz\Big) dy$$
$$\le \Big(\int_0^M |\bar{g}(x)|^2 dx\Big)^{1/2} + \Big(\int_0^M |\bar{g}'(x)|^2 dx\Big)^{1/2}$$
$$= 1 + \Big(\int_0^M |\bar{g}'(x)|^2 dx\Big)^{1/2}.$$

Notice that this inequality holds even $x+1 \notin [0, M]$ in which case we integrate on $[x - 1, x]$ instead on $[x, x + 1]$.

Consequently,

$$\Big(\int_0^M |\bar{g}(x)|^{2p} dx\Big)^{1/p} \le \alpha\Big(\int_{-\infty}^{\infty} |f(x)|^{2p} dx\Big)^{1/p} \tag{4.2.21}$$
$$+ \big((2M)^{-1/2}\big)^{1/p}\Big\{1 + \Big(\int_0^M |\bar{g}'(x)|^2 dx\Big)^{1/2}\Big\}^{2/q}.$$

Combining (4.2.20) and (4.2.21),

$$\Big(\int_0 |\bar{g}(x)|^{2p} dx\Big)^{1/p} - \frac{1-\epsilon}{2}\int_0^M |\bar{g}'(x)|^2 dx$$
$$-\frac{1}{\sqrt{M}}\Big(\int_0^M |\bar{g}'(x)|^2 dx\Big)^{1/2} - \big((2M)^{-1/2}\big)^{1/p}\Big\{1 + \Big(\int_0^M |\bar{g}'(x)|^2 dx\Big)^{1/2}\Big\}^{2/q}$$
$$\le \frac{1-\epsilon}{2M} + \alpha\Big\{\Big(\int_{-\infty}^{\infty} |f(x)|^{2p} dx\Big)^{1/p} - \frac{1-\epsilon}{2}\int_{-\infty}^{\infty} |f'(x)|^2 dx\Big\}$$
$$\le \frac{1-\epsilon}{2M} + \alpha\Big(\frac{1}{1-\epsilon}\Big)^{\frac{p-1}{p+1}} J$$

where the last step follows partially from the substitution

$$f(x) = \Big(\frac{1}{1-\epsilon}\Big)^{\frac{p}{2(p+1)}} h\bigg(\Big(\frac{1}{1-\epsilon}\Big)^{\frac{p}{(p+1)}} x\bigg).$$

Since $q \le 2$, there is a sufficiently large $M = M(\epsilon)$ such that

$$\frac{1}{\sqrt{M}} x + \big((2M)^{-1/2}\big)^{1/p}(1+x)^q \le \frac{\epsilon}{2}(1+x^2)$$

for all $x \ge 0$. For such M,

$$\begin{aligned}
&\bigg(\int_0 |\bar{g}(x)|^{2p} dx\bigg)^{1/p} - \frac{1}{2}\int_0^M |\bar{g}'(x)|^2 dx \\
&\le \frac{\epsilon}{2} + \frac{1-\epsilon}{2M} + \alpha\Big(\frac{1}{1-\epsilon}\Big)^{\frac{p-1}{p+1}} J.
\end{aligned}$$

We may take $M \ge \epsilon^{-1}$. Notice that $\alpha \le 1$. Finally, (4.2.17) follows from the inequality in (4.2.18) which gives that

$$\int_0^M |\bar{g}'(x)|^2 dx \le \int_{-\infty}^{\infty} |g'(x)|^2 dx.$$

□

In the case $d \ge 2$, the local time $L(t,x)$ no longer exists. We may introduce

$$L(t,x,\epsilon) = \int_0^t p_\epsilon\big(W(s) - x\big) ds$$

instead. With some slight modification on the proof of (4.2.2), we can prove (the reader is also refereed to Theorem 3.1 of [24] for a detailed proof) the following

Theorem 4.2.3 *For each $d \ge 1$, $p > 1$ and $\epsilon, \theta > 0$,*

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\bigg\{\theta\bigg(\int_{\mathbb{R}^d} L^p(t,x,\epsilon) dx\bigg)^{1/p}\bigg\} \\
&= \sup_{g \in \mathcal{F}_d} \bigg\{\theta\bigg(\int_{\mathbb{R}^d} \big[g(x,\epsilon)\big]^{2p} dx\bigg)^{1/p} - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx\bigg\}
\end{aligned}$$

where $\mathcal{F}_d$ is defined in (4.1.22) and

$$g(x,\epsilon) = \bigg\{\int_{\mathbb{R}^d} p_\epsilon(y) g^2(x-y) dx\bigg\}^{1/2} \qquad x \in \mathbb{R}^d.$$

4.3 Two dimensional case

Recall that in Section 2.4 we constructed the renormalized self-intersection local time $\gamma\big([0,t]^2_<\big)$ (formally written in (2.4.1)) run by a 2-dimensional Brownian motion $W(t)$. The discussion naturally leads to the "renormalized" polymer models

$$\widehat{\mathbb{P}}_\lambda(A) = \widehat{C}_\lambda^{-1}\mathbb{E}\bigg(\exp\Big\{\lambda\gamma\big([0,1]^2_<\big)\Big\}1_{\{W(\cdot)\in A\}}\bigg) \qquad A\subset C\big\{[0,1],\mathbb{R}^2\big\} \quad (4.3.1)$$

$$\widetilde{\mathbb{P}}_\lambda(A) = \widetilde{C}_\lambda^{-1}\mathbb{E}\bigg(\exp\Big\{-\lambda\gamma\big([0,1]^2_<\big)\Big\}1_{\{W(\cdot)\in A\}}\bigg) \quad A\subset C\big\{[0,1],\mathbb{R}^2\big\} \quad (4.3.2)$$

where

$$\widehat{C}_\lambda = \mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]^2_<\big)\Big\} \;\text{ and }\; \widetilde{C}_\lambda = \mathbb{E}\exp\Big\{-\lambda\gamma\big([0,1]^2_<\big)\Big\}$$

are normalizers.

In view of Theorem 2.4.2, $\widetilde{C}_\lambda<\infty$ for all $\lambda>0$. In the term of physics, it shows that there is no phase transition in the self-repelling polymer model given in (4.3.2). On the other hand, we shall show that there is a $\lambda_0>0$ such that [123] that $\widehat{C}_\lambda=\infty$ for sufficiently large $\lambda>0$. An important question is to find the $\lambda_0>0$ such that

$$\mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]^2_<\big)\Big\}\begin{cases} <\infty & \lambda<\lambda_0 \\ \\ =\infty & \lambda>\lambda_0. \end{cases} \tag{4.3.3}$$

In physics, the value λ_0 is critical to the "melt-down" of a self-attracting polymer. We shall identify λ_0 in terms of Gagliardo-Nirenberg constant.

Theorem 4.3.1

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\gamma\big([0,1]^2_<\big)\ge t\Big\} = -\kappa(2,2)^{-4}.$$

Here we recall that $\kappa(2,2)>0$ *is the best constant of the Gagliardo-Nirenberg inequality*

$$||f||_4 \le C\sqrt{||\nabla f||_2}\sqrt{||f||_2} \qquad f\in W^{1,2}(\mathbb{R}^2) \tag{4.3.4}$$

where

$$W^{1,2}(\mathbb{R}^2) = \big\{f\in\mathcal{L}^2(\mathbb{R}^2);\;\; \nabla f\in\mathcal{L}^2(\mathbb{R}^d)\big\}.$$

Proof. Let $0 \le \lambda_0 \le \infty$ be defined by (4.3.3). Then

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{\gamma\big([0,1]^2_<\big) \ge t\Big\} = -\lambda_0.$$

By Theorem 2.4.2 $\lambda_0 > 0$. To establish the upper bound, we need to show that $\lambda_0 \ge \kappa(2,2)^{-4}$. For this purpose we may assume that $\lambda_0 < \infty$.

Consider the decomposition

$$\gamma\big([0,1]^2_<\big) = \gamma\big([0,1/2]^2_<\big) + \gamma\big([1/2,1]^2_<\big) + \gamma\big([0,1/2]\times[1/2,1]\big).$$

It gives that for each $\epsilon > 0$,

$$\begin{aligned} -\lambda_0 \le \max\bigg\{ &\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{\gamma\big([0,1/2]^2_<\big) + \gamma\big([1/2,1]^2_<\big) \ge \frac{1+\epsilon}{2}t\Big\}, \\ &\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{\gamma\big([0,1/2]\times[1/2,1]\big) \ge \frac{1-\epsilon}{2}t\Big\}\bigg\}. \end{aligned}$$

By Proposition 2.3.4,

$$\gamma\big([0,1/2]\times[1/2,1]\big) \overset{d}{=} \frac{1}{2}\Big\{\alpha\big([0,1]^2\big) - \mathbb{E}\alpha\big([0,1]^2\big)\Big\}.$$

Taking $d = p = 2$ in (3.3.4),

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{\alpha\big([0,1]^2\big) \ge t\Big\} = -\kappa(2,2)^{-4}.$$

Consequently,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{\gamma\big([0,1/2]\times[1/2,1]\big) \ge \frac{1-\epsilon}{2}t\Big\} = -(1-\epsilon)\kappa(2,2)^{-4}.$$

Observe that $\gamma\big([0,1/2]^2_<\big)$ and $\gamma\big([1/2,1]^2_<\big)$ are independent and have the same law as $(1/2)\gamma\big([0,1]^2_<\big)$ Thus,

$$\mathbb{E}\exp\Big\{\lambda\Big(\gamma\big([0,1/2]^2_<\big) + \gamma\big([1/2,1]^2_<\big)\Big)\Big\} = \Big[\mathbb{E}\exp\Big\{\frac{\lambda}{2}\gamma\big([0,1]^2\big)\Big\}\Big]^2 < \infty$$

for every $\lambda < 2\lambda_0$. Hence,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{\gamma\big([0,1/2]^2_<\big) + \gamma\big([1/2,1]^2_<\big) \ge \frac{1+\epsilon}{2}t\Big\} \le -(1+\epsilon)\lambda_0.$$

Summarizing our argument,

$$-\lambda_0 \le \max\Big\{-(1+\epsilon)\lambda_0,\ \ -(1-\epsilon)\kappa(2,2)^{-4}\Big\}.$$

By the fact that $\lambda_0 > 0$, we must have $\lambda_0 \geq (1-\epsilon)\kappa(2,2)^{-4}$. Letting $\epsilon \to 0^+$ we have proved the upper bound

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{\gamma\big([0,1]^2_<\big) \geq t\Big\} \leq -\kappa(2,2)^{-4}. \tag{4.3.5}$$

To establish the lower bound, recall that

$$L(t,x,\epsilon) = \int_0^t p_\epsilon\big(W(s)-x\big)ds$$

and notice that

$$\int_{\mathbb{R}^2} L^2(t,x,\epsilon)dx = 2\iint_{\{0\leq r<s\leq t\}} p_{2\epsilon}\big(W(r)-W(t)\big)drds.$$

By Theorem 4.2.6 (with $p=2$) for any $\theta > 0$,

$$\begin{aligned}&\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{\theta\bigg(\iint_{\{0\leq r<s\leq t\}} p_{2\epsilon}\big(W(r)-W(t)\big)drds\bigg)^{1/2}\right\}\\ &= \sup_{g\in\mathcal{F}_2}\left\{\frac{\theta}{\sqrt{2}}\bigg(\int_{\mathbb{R}^2} g^4(x,\epsilon)dx\bigg)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx\right\}.\end{aligned}$$

Let $t = n$ be positive integer and write

$$\begin{aligned}&\iint_{\{0\leq r<s\leq n\}} p_{2\epsilon}\big(W(r)-W(t)\big)drds \\ &= \sum_{k=1}^{n-1} \beta_\epsilon\big([0,k]\times[k,k+1]\big) + \sum_{k=1}^{n} \beta_\epsilon\big([k-1,k]^2_<\big).\end{aligned} \tag{4.3.6}$$

Here we recall that

$$\beta_\epsilon(A) = \int_A p_{2\epsilon}\big(W(r)-W(s)\big)drds,$$

$$\beta(A) = \int_A \delta_0\big(W(r)-W(s)\big)drds$$

are the random measures discussed in Section 2.3 (with $p = d = 2$). By Theorem 2.3.2 and a treatment similar to the one used in Proposition 2.3.4 we conclude that for any fixed n,

$$\sum_{k=1}^{n-1} \beta_\epsilon\big([0,k]\times[k,k+1]\big) \overset{\epsilon\to 0^+}{\longrightarrow} \sum_{k=1}^{n-1} \beta\big([0,k]\times[k,k+1]\big)$$

in $\mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ for all $m > 0$.

Notice that the second term on the right hand side of (4.3.6) is bounded by $C_\epsilon n$ for some constant $C_\epsilon > 0$. So we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp\left\{\theta\bigg(\sum_{k=1}^{n-1} \beta_\epsilon\big([0,k]\times[k,k+1]\big)\bigg)^{1/2}\right\}$$
$$= \sup_{g\in\mathcal{F}_2}\left\{\frac{\theta}{\sqrt{2}}\bigg(\int_{\mathbb{R}^2} g^4(x,\epsilon)dx\bigg)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx\right\}.$$

By Lemma 1.2.5, this leads to

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!}\left\{\mathbb{E}\bigg[\sum_{k=1}^{n-1} \beta_\epsilon\big([0,k]\times[k,k+1]\big)\bigg]^m\right\}^{1/2} \tag{4.3.7}$$
$$= \frac{1}{2}\sup_{g\in\mathcal{F}_2}\left\{\sqrt{2}\theta\bigg(\int_{\mathbb{R}^2} g^4(x,\epsilon)dx\bigg)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx\right\}.$$

We now claim that for any $\epsilon > 0$, integers $m, n \ge 1$,

$$\mathbb{E}\bigg[\sum_{k=1}^{n-1} \beta_\epsilon\big([0,k]\times[k,k+1]\big)\bigg]^m \le \mathbb{E}\bigg[\sum_{k=1}^{n-1} \beta\big([0,k]\times[k,k+1]\big)\bigg]^m. \tag{4.3.8}$$

Indeed, write

$$D = \bigcup_{k=1}^{n-1} [0,k]\times[k,k+1].$$

By Fourier inversion (Theorem B.1, Appendix),

$$p_{2\epsilon}(x) = (2\pi)^{-2}\int_{\mathbb{R}^2} e^{-i\lambda\cdot x}\exp\big\{-\epsilon|\lambda|^2\big\}d\lambda.$$

Thus,

$$\sum_{k=1}^{n-1} \beta_\epsilon\big([0,k]\times[k,k+1]\big) = \int_D drds p_{2\epsilon}\big(W(r)-W(s)\big)drds$$
$$= \int_{\mathbb{R}^2} d\lambda \exp\big\{-\epsilon|\lambda|^2\big\}\int_D drds \exp\big\{-i\lambda\cdot\big(W(r)-W(s)\big)\big\}.$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big)\Big]^m\\
&=\int_{(\mathbb{R}^2)^m} d\lambda_1\cdots d\lambda_m \exp\Big\{-\epsilon\sum_{k=1}^m|\lambda_k|^2\Big\}\\
&\times\int_{D^m} dr_1ds_1\cdots dr_m ds_m \exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^m\lambda_k\cdot\big(W(r_k)-W(s_k)\big)\Big)\Big\}.
\end{aligned}$$

Therefore, for any $0<\epsilon'<\epsilon$,

$$\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta_\epsilon\big([0,k]\times[k,k+1]\big)\Big]^m \le \mathbb{E}\Big[\sum_{k=1}^{n-1}\beta_{\epsilon'}\big([0,k]\times[k,k+1]\big)\Big]^m$$

which leads to (4.3.8) by letting $\epsilon'\to 0$.

By (4.3.7) and (4.3.8),

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big\{\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\Big]^m\Big\}^{1/2}\\
&\ge\frac{1}{2}\sup_{g\in\mathcal{F}_2}\Big\{\sqrt{2}\theta\Big(\int_{\mathbb{R}^2}g^4(x,\epsilon)dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big\}.
\end{aligned}$$

When $\epsilon\to 0^+$, the right hand side converges to

$$\frac{1}{2}\sup_{g\in\mathcal{F}_2}\Big\{\sqrt{2}\theta\Big(\int_{\mathbb{R}^2}|g(x)|^4dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big\}=\frac{1}{2}\theta^2\kappa(2,2)^4$$

where the equality follows from Theorem C.1, Appendix (with $d=p=2$). Thus,

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big\{\mathbb{E}\Big[\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\Big]^m\Big\}^{1/2}\\
&\ge\frac{1}{2}\theta^2\kappa(2,2)^4\qquad(\theta>0).
\end{aligned}\tag{4.3.9}$$

Notice that (4.3.9) along is not enough for the lower bound of the large deviation. In the following effort, we strengthen (4.3.9) into an equality.

By scaling property, (4.3.5) can be rewritten as

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\gamma\big([0,n]_<^2\big)\ge\lambda n^2\Big\}\le-\lambda\kappa(2,2)^{-4}\qquad(\lambda>0).$$

Similar to the decomposition (4.3.6),

$$\gamma\big([0,n]_<^2\big)=\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)-\sum_{k=1}^{n-1}\mathbb{E}\beta\big([0,k]\times[k,k+1]\big)\tag{4.3.10}$$
$$+\sum_{k=1}^{n}\gamma\big([k-1,k]_<^2\big).$$

Notice that

$$\sum_{k=1}^{n-1}\mathbb{E}\beta\big([0,k]\times[k,k+1]\big)=\frac{1}{2\pi}\sum_{k=1}^{n-1}\int_0^k\int_k^{k+1}\frac{1}{s-r}dsdr\tag{4.3.11}$$
$$=\frac{1}{2\pi}\sum_{k=1}^{n-1}\Big\{\log(k+1)-\log k\Big\}=\frac{1}{2\pi}n\log n.$$

The random variables $\gamma\big([0,n]_<^2\big)$ $(k=1,2\cdots)$ forms an i.i.d. exponentially integrable sequence. By a standard application of Chebyshev inequality,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\Big|\sum_{k=1}^{n}\gamma\big([k-1,k]_<^2\big)\Big|\ge\epsilon n^2\Big\}=-\infty.\tag{4.3.12}$$

Summarizing our argument,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\ge\lambda n^2\Big\}\le-\lambda\kappa(2,2)^{-4}.$$

By Lemma 1.2.8 (part (2)) and Lemma 1.2.9, this leads to

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}\bigg\{\mathbb{E}\bigg[\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\bigg]^m\bigg\}^{1/2}$$
$$\le\sup_{\lambda>0}\Big\{\theta\lambda^{1/2}-\frac{1}{2}\lambda\kappa(2,2)^{-4}\Big\}=\frac{1}{2}\theta^2\kappa(2,2)^4.$$

Combining this with (4.3.9) gives

$$\lim_{n\to\infty}\frac{1}{n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}\bigg\{\mathbb{E}\bigg[\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\bigg]^m\bigg\}^{1/2}=\frac{1}{2}\theta^2\kappa(2,2)^4.$$

By Theorem 1.2.6,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\ge\lambda n^2\Big\}=-\lambda\kappa(2,2)^{-4}.$$

We bring this back to the decomposition (4.3.10). By (4.3.11) and (4.3.12),

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{\gamma\big([0,n]_<^2\big)\ge\lambda n^2\Big\}=-\lambda\kappa(2,2)^{-4}$$

which ends the proof. □

By comparing Theorem 4.3.1 with (3.3.4) (with $d=p=2$), one can see that the intersection local time $\alpha\big([0,1/2]^2\big)$ obeys the same large deviation principle as $\gamma\big([0,1]_<^2\big)$. This observation is crucial in our proof the upper bound part of Theorem 4.3.1, since $\alpha\big([0,1/2]^2\big)\overset{d}{=}2^{-1}\alpha\big([0,1]^2\big)$ so the tail of $\alpha\big([0,1/2]^2\big)$ is given by Theorem 3.3.2. We try to turn our mathematical argument into the following intuitive explanation:

Cut the Brownian path $\Big\{\big(t,W(t)\big);\ 0\le t\le 1\Big\}$ into two paths at the middle point $\big(1/2,W(1/2)\big)$ and shift the whole system so the cutting point $\big(1/2,W(1/2)\big)$ becomes the origin after shifting. If we reverse the direction of the path before the time $t=1/2$, then the resulted two paths are trajectories of two independent Brownian motions running up to time 1/2.

The total self-intersection of the original path $\Big\{\big(t,W(t)\big);\ 0\le t\le 1\Big\}$ is the sum of intersection within each sub-path and the intersection between two sub-paths. When $d=2$, the first type intersection out-numbers the second type intersection (that is the main reason why $\beta\big([0,1]_<^2\big)=\infty$ (Proposition 2.3.6). The renormalization subdues the short range intersection so that these two intersections are comparable. By chance, the ratio is 1 to 1 here in the sense of large deviations. That is to say, about half of $\gamma\big([0,1]^p\big)$ is made of a random quantity distributed as $\alpha\big([0,1/2]^2\big)$. As for another half, it is equal to the sum of the (renormalized) self-intersection local times of two independent paths and each of them can be analyzed in the way proposed above.

The proportion between these two types of intersection varies in different settings. From later development, we shall see that finding this proportion is an important part of establishing the large deviations related to self intersections.

The following theorem shows a completely different tail asymptotic behavior.

Theorem 4.3.2 *There is a constant* $0<L<\infty$ *such that for any* $\theta>0$

$$\lim_{t\to\infty}t^{-2\pi\theta}\log\mathbb{P}\Big\{-\gamma\big([0,1]_<^2\big)\ge\theta\log t\Big\}=-L.$$

Proof. One needs only to prove Theorem 4.3.2 in the case $\theta = (2\pi)^{-1}$. That is,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{-\gamma\big([0,1]^2_<\big)\ge(2\pi)^{-1}\log t\Big\} = -L. \tag{4.3.13}$$

for some $0 < L < \infty$. Indeed, the general statement will follows from (4.3.2) if we substitute t by $t^{2\pi\theta}$.

Recall ((2.4.10)) that

$$Z_t = -\frac{1}{2\pi}t\log t - \gamma\big([0,t]^2_<\big) \qquad t\ge 0.$$

By (2.4.11), for any $s,t>0$,

$$\mathbb{E}\exp\Big\{2\pi Z_{s+t}\Big\}\le\mathbb{E}\exp\Big\{2\pi Z_s\Big\}\mathbb{E}\exp\Big\{2\pi Z_t\Big\}$$

or,

$$\begin{aligned}&(s+t)^{-(s+t)}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,s+t]^2_<\big)\Big\}\\ &\le\bigg(s^{-s}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,s]^2_<\big)\Big\}\bigg)\bigg(t^{-t}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,t]^2_<\big)\Big\}\bigg).\end{aligned}$$

By Lemma 1.3.1, the limit

$$\lim_{t\to\infty}\frac{1}{t}\log\bigg(t^{-t}\mathbb{E}\exp\Big\{-2\pi\gamma\big([0,t]^2_<\big)\Big\}\bigg) = A \tag{4.3.14}$$

exists with $A<\infty$. By Lemma 4.3.3 below, $A>-\infty$. Let $t=n$ be integer. By Stirling formula and by scaling,

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\mathbb{E}\exp\Big\{-2\pi n\gamma\big([0,1]^2_<\big)\Big\} = A+1.$$

Finally, applying Theorem 1.2.7 to the non-negative random variable

$$Y = \exp\Big\{-2\pi\gamma\big([0,1]^2_<\big)\Big\}$$

leads to (4.3.13) with

$$L = \exp\{-1-A\}. \tag{4.3.15}$$

□

We ends this section by establishing the following lemma.

Lemma 4.3.3

$$\liminf_{n\to\infty}\frac{1}{n}\log\left(n^{-n}\mathbb{E}\exp\left\{-2\pi\gamma\big([0,n]^2_<\big)\right\}\right)>-\infty.$$

Proof. By (4.3.10) and (4.3.11)

$$\begin{aligned}
&n^{-n}\mathbb{E}\exp\left\{-2\pi\gamma\big([0,n]^2_<\big)\right\}\\
&=\exp\left\{-2\pi\sum_{k=1}^{n}\gamma\big([k-1,k]^2_<\big)-2\pi\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\right\}\\
&\ge e^{-2\pi Mn}\mathbb{E}\exp\left\{-2\pi\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\right\}\\
&-e^{-2\pi Mn}\mathbb{P}\Big\{\sum_{k=1}^{n}\gamma\big([k-1,k]^2_<\big)\ge Mn\Big\}.
\end{aligned}$$

By Chebyshev inequality, for any $0<\lambda<\kappa(2,2)^{-4}$,

$$\mathbb{P}\Big\{\sum_{k=1}^{n}\gamma\big([k-1,k]^2_<\big)\ge Mn\Big\}\le e^{-n\lambda M}\left(\mathbb{E}\exp\left\{\lambda\gamma\big([0,1]^2_<\big)\right\}\right)^n$$

and the right hand side can be exponentially small to any requested level by choosing sufficiently large M. To establish $A>-\infty$, therefore, we need only to establish the lower bound

$$\mathbb{E}\exp\left\{-2\pi\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\right\}\ge c_1^n \tag{4.3.16}$$

for some constant $c_1>0$ and sufficiently large n.

Let the 1-dimensional Brownian motion $W_o(t)$ be the first component of $W(t)$ and write

$$D_n=\Big\{\sup_{0\le s\le n}|W_o(s)-s|\le\delta\Big\}$$

where $0<\delta<1/2$ is fixed. Notice that on D_n,

$$\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)=\sum_{k=1}^{n-1}\beta\big([k-1,k]\times[k,k+1]\big).$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\exp\Big\{-2\pi\sum_{k=1}^{n-1}\beta\big([0,k]\times[k,k+1]\big)\Big\} \qquad (4.3.17)\\
&\geq \mathbb{E}\Big[\exp\Big\{-2\pi\sum_{k=1}^{n-1}\beta\big([k-1,k]\times[k,k+1]\big)\Big\}1_{D_n}\Big]\\
&\geq e^{-2\pi Mn}\Big\{\mathbb{P}(D_n)-\mathbb{P}\Big\{\sum_{k=1}^{n-1}\beta\big([k-1,k]\times[k,k+1]\big)\geq Mn\Big\}\Big\}.
\end{aligned}$$

Write

$$\begin{aligned}
&\sum_{k=1}^{n-1}\beta\big([k-1,k]\times[k,k+1]\big)\\
&=\sum_k\beta\big([2(k-1),2k-1]\times[2k-1,2k]\big)\\
&+\sum_k\beta\big([2k-1,2k]\times[2k,2k+1]\big).
\end{aligned}$$

By triangular inequality,

$$\begin{aligned}
&\mathbb{P}\Big\{\sum_{k=1}^{n-1}\beta\big([k-1,k]\times[k,k+1]\big)\geq Mn\Big\}\\
&\leq \mathbb{P}\Big\{\sum_k\beta\big([2(k-1),2k-1]\times[2k-1,2k]\big)\geq 2^{-1}Mn\Big\}\\
&+\mathbb{P}\Big\{\sum_k\beta\big([2k-1,2k]\times[2k,2k+1])\big)\geq 2^{-1}Mn\Big\}.
\end{aligned}$$

By the fact that the sequence

$$\beta\big([2(k-1),2k-1]\times[2k-1,2k]\big) \qquad k=1,\cdots$$

is i.i.d. with common distribution same as $\alpha([0,1]^2)$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sum_k\beta\big([2(k-1),2k-1]\times[2k-1,2k]\big)\geq 2^{-1}Mn\Big\}\\
&\leq e^{-2^{-1}n\lambda M}\Big(\mathbb{E}\exp\Big\{\lambda\alpha\big([0,1]^2\big)\Big\}\Big)^n
\end{aligned}$$

where $\lambda>0$ is chosen in order that

$$\mathbb{E}\exp\Big\{\lambda\alpha\big([0,1]^2\big)\Big\}<\infty.$$

Similarly,

$$\mathbb{P}\Big\{\sum_k \beta\big([2k-1,2k]\times[2k,2k+1]\big)) \geq 2^{-1}Mn\Big\}$$
$$\leq e^{-2^{-1}n\lambda M}\bigg(\mathbb{E}\exp\Big\{\lambda\alpha\big([0,1]^2\big)\Big\}\bigg)^n.$$

Thus, by triangular inequality

$$\mathbb{P}\Big\{\sum_{k=1}^{n-1}\beta\big([k-1,k]\times[k,k+1]\big)\geq Mn\Big\}$$
$$\leq 2e^{-2^{-1}n\lambda M}\bigg(\mathbb{E}\exp\Big\{\lambda\alpha\big([0,1]^2\big)\Big\}\bigg)^n.$$

On the other hand,

$$D_n = \Big\{\sup_{0\leq s\leq n-1}|W_o(s)-s|\leq\delta\Big\}$$
$$\cap\Big\{\sup_{0\leq s\leq 1}\big|\big(W_o(n-1+s)-W_o(n-1)\big)-s+\big(W_o(n-1)-(n-1)\big)\big|\leq\delta\Big\}.$$

By independence of increments,

$$\mathbb{P}(D_n)\geq\mathbb{P}(D_{n-1})\inf_{|x|\leq\delta}\mathbb{P}_x\Big\{\sup_{0\leq s\leq 1}\big|W_o(s)-s\big|\leq\delta\Big\}.$$

Repeating our argument,

$$\mathbb{P}(D_n)\geq\bigg(\inf_{|x|\leq\delta}\mathbb{P}_x\Big\{\sup_{0\leq s\leq 1}\big|W_o(s)-s\big|\leq\delta\Big\}\bigg)^n = c_2^n,$$

where

$$c_2\equiv\inf_{|x|\leq\delta}\mathbb{P}_x\Big\{\sup_{0\leq s\leq 1}\big|W_o(s)-s\big|\leq\delta\Big\}>0.$$

Thus, one can take $M>0$ sufficiently large so that

$$\mathbb{P}\Big\{\sum_{k=1}^{n-1}\beta\big([k-1,k]\times[k,k+1]\big)\geq Mn\Big\}\leq\frac{1}{2}\mathbb{P}(D_n).$$

Hence, (4.3.16) follows from (4.3.17). $\square$

4.4 Applications to LIL

In this section we apply the large deviations established in Section 4.2 and Section 4.3 to the laws of the iterated logarithm for self-intersection local times.

Theorem 4.4.1 *Let $L(t,x)$ be the local time of an 1-dimensional Brownian motion. For any $p>1$,*

$$\limsup_{t\to\infty} t^{-\frac{p+1}{2}}(\log\log t)^{-\frac{p-1}{2}}\int_{-\infty}^{\infty} L^p(t,x)dx$$
$$= 2^{\frac{p+1}{2}}(p-1)^{\frac{p-1}{2}}(p+1)^{\frac{p-3}{2}}B\Big(\frac{1}{p-1},\frac{1}{2}\Big)^{-(p-1)} \qquad a.s.$$

Proof. By the scaling (4.2.1), Theorem 4.2.1 can be rewritten as

$$\lim_{t\to\infty}\frac{1}{\log\log t}\log\mathbb{P}\Big\{\int_{-\infty}^{\infty} L^p(t,x)dx \ge \lambda t^{\frac{p+1}{2}}(\log\log t)^{\frac{p-1}{2}}\Big\}$$
$$= -\lambda^{\frac{2}{p-1}}\frac{1}{4(p-1)}\Big(\frac{p+1}{2}\Big)^{\frac{3-p}{p-1}}B\Big(\frac{1}{p-1},\frac{1}{2}\Big).$$

Notice that

$$\lambda_0 \equiv 2^{\frac{p+1}{2}}(p-1)^{\frac{p-1}{2}}(p+1)^{\frac{p-3}{2}}B\Big(\frac{1}{p-1},\frac{1}{2}\Big)^{-(p-1)}$$

solves the equation

$$\lambda^{\frac{2}{p-1}}\frac{1}{4(p-1)}\Big(\frac{p+1}{2}\Big)^{\frac{3-p}{p-1}}B\Big(\frac{1}{p-1},\frac{1}{2}\Big) = 1.$$

A standard approach similar to the one used for (3.3.10) gives the upper bound

$$\limsup_{t\to\infty} t^{-\frac{p+1}{2}}(\log\log t)^{-\frac{p-1}{2}}\int_{-\infty}^{\infty} L^p(t,x)dx \le \lambda_0 \qquad a.s.$$

To establish the lower bound, write $t_k = k^k$ $(k=1,2\cdots)$ and let $\lambda' < \lambda_0$ be arbitrary. We have

$$\sum_{k=1}^{\infty}\mathbb{P}\Big\{\int_{-\infty}^{\infty}\big[L(t_{k+1},x)-L(t_k,x)\big]^p dx \ge \lambda' t_{k+1}^{\frac{p+1}{2}}(\log\log t_{k+1})^{\frac{p-1}{2}}\Big\}$$
$$= \sum_{k=1}^{\infty}\mathbb{P}\Big\{\int_{-\infty}^{\infty} L^p(t_{k+1}-t_k,x)dx \ge \lambda' t_{k+1}^{\frac{p+1}{2}}(\log\log t_{k+1})^{\frac{p-1}{2}}\Big\} = \infty.$$

Notice that

$$\int_{-\infty}^{\infty} \big[L(t_{k+1},x)-L(t_k,x)\big]^p dx \qquad k=1,2,\cdots$$

is an independent sequence. By Borel-Cantelli lemma

$$\limsup_{k\to\infty} t_{k+1}^{-\frac{p+1}{2}}(\log\log t_{k+1})^{-\frac{p-1}{2}}\int_{-\infty}^{\infty}\big[L(t_{k+1},x)-L(t_k,x)\big]^p dx \ge \lambda' \qquad a.s.$$

which leads to

$$\limsup_{t\to\infty} t^{-\frac{p+1}{2}}(\log\log t)^{-\frac{p-1}{2}}\int_{-\infty}^{\infty} L^p(t,x)dx \ge \lambda' \qquad a.s.$$

Letting $\lambda'\to\lambda_0^-$ on the right hand side gives the lower bound. □

Theorem 4.4.2 *Let $\gamma\big([0,t]_<^2\big)$ be the renormalized self-intersection local time of a planar Brownian motion $W(t)$.*

$$\limsup_{t\to\infty}\frac{1}{t\log\log t}\gamma\big([0,t]_<^2\big)=\kappa(2,2)^4 \qquad a.s. \tag{4.4.1}$$

$$\liminf_{t\to\infty}\frac{1}{t\log\log\log t}\gamma\big([0,t]_<^2\big)=-\frac{1}{2\pi} \qquad a.s. \tag{4.4.2}$$

Proof. Let $\theta>0$ be fixed but arbitrary and write $t_n=\theta^n$. By Theorem 4.3.1 and Borel-Cantelli lemma,

$$\limsup_{n\to\infty}\frac{1}{t_n\log\log t_n}\gamma\big([0,t_n]_<^2\big)\le\kappa(2,2)^4 \qquad a.s. \tag{4.4.3}$$

Given $\epsilon>0$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t_{n-1}\le t\le t_n}\big|\gamma\big([0,t]_<^2\big)-\gamma\big([0,t_{n-1}]_<^2\big)\big|\ge\epsilon t_n\log\log t_n\Big\}\\
&\le\mathbb{P}\Big\{\sup_{\substack{|t-s|\le(\theta-1)t_n\\ s,t\in[0,t_n]}}\big|\gamma\big([0,t]_<^2\big)-\gamma\big([0,s]_<^2\big)\big|\ge\epsilon t_n\log\log t_n\Big\}\\
&=\mathbb{P}\Big\{\sup_{\substack{|t-s|\le(\theta-1)\\ s,t\in[0,1]}}\big|\gamma\big([0,t]_<^2\big)-\gamma\big([0,s]_<^2\big)\big|\ge\epsilon\log\log t_n\Big\}.
\end{aligned}$$

where the last step follows from scaling. By (2.4.12), therefore, one can have

$$\sum_n\mathbb{P}\Big\{\sup_{t_{n-1}\le t\le t_n}\big|\gamma\big([0,t]_<^2\big)-\gamma\big([0,t_{n-1}]_<^2\big)\big|\ge\epsilon t_n\log\log t_n\Big\}<\infty$$

by making θ close to 1 enough. By Borel-Cantelli lemma,

$$\limsup_{n\to\infty}\frac{1}{t_n\log\log t_n}\sup_{t_{n-1}\le t\le t_n}\left|\gamma\big([0,t]_<^2\big)-\gamma\big([0,t_{n-1}]_<^2\big)\right|\le\epsilon \qquad a.s.$$

Combining this with (4.4.3) we establish the upper bound

$$\limsup_{t\to\infty}\frac{1}{t\log\log t}\gamma\big([0,t]_<^2\big)\le\kappa(2,2)^4 \qquad a.s. \tag{4.4.4}$$

Write $K_n=[\log\log t_n]$ and $s_n=t_n/K_n$. For $(l-1)s_n\le t\le ls_n$

$$\begin{aligned}-\gamma\big([0,t]_<^2\big)&\le-\sum_{k=1}^{l-1}\gamma\big([(k-1)s_n,ks_n]_<^2\big)-\gamma\big([(l-1)s_n,s]_<^2\big)\\&\quad+\sum_{k=1}^{K_n-1}\mathbb{E}\beta\big([0,(k-1)s_n]\times[ks_n,(k+1)s_n]\big)\\&\le\sum_{k=1}^{K_n}\sup_{(k-1)s_n\le s\le ks_n}\Big\{-\gamma\big([(k-1)s_n,s]_<^2\big)\Big\}\\&\quad+\sum_{k=1}^{K_n-1}\mathbb{E}\beta\big([0,(k-1)s_n]\times[ks_n,(k+1)s_n]\big).\end{aligned}$$

The second term on the right hand side is equal to

$$\begin{aligned}&\frac{1}{2\pi}\sum_{k=1}^{K_n-1}\Big\{(k+1)s_n\log\big((k+1)s_n\big)-ks_n\log(ks_n)-s_n\log s_n\Big\}\\&=\frac{1}{2\pi}K_ns_n\log K_n\le\frac{1}{2\pi}t_n\log\log t_n.\end{aligned}$$

Thus, for any $\epsilon>0$,

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\le t_n}\Big\{-\gamma\big([0,t]_<^2\big)\Big\}\ge(1+\epsilon)t_n\log\log\log t_n\Big\}\\&\le\mathbb{P}\Big\{\sum_{k=1}^{K_n}\sup_{(k-1)s_n\le s\le ks_n}\Big\{-\gamma\big([(k-1)s_n,s]_<^2\big)\Big\}\ge\epsilon t_n\log\log\log t_n\Big\}.\end{aligned}$$

The random sequence

$$\sup_{(k-1)s_n\le s\le ks_n}\Big\{-\gamma\big([(k-1)s_n,s]_<^2\big)\Big\} \qquad k=1,2,\cdots$$

is an i.i.d. with the common distribution same as

$$s_n\sup_{0\le s\le 1}\Big\{-\gamma\big([0,s]_<^2\big)\Big\}.$$

By Chebyshev inequality, therefore,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\le t_n}\Big\{-\gamma\big([0,t]_<^2\big)\Big\}\ge(1+\epsilon)t_n\log\log\log t_n\Big\}\\
&\le\exp\Big\{-\lambda\epsilon K_n\log K_n\Big\}\Big(\mathbb{E}\exp\Big\{\lambda\sup_{0\le s\le 1}\Big\{-\gamma\big([0,s]_<^2\big)\Big\}\Big\}\Big)^{K_n}
\end{aligned}$$

where, by (2.4.13), we can take $\lambda>0$ small enough so

$$\mathbb{E}\exp\Big\{\lambda\sup_{0\le s\le 1}\Big\{-\gamma\big([0,s]_<^2\big)\Big\}\Big\}<\infty.$$

Therefore,

$$\sum_n\mathbb{P}\Big\{\sup_{t\le t_n}\Big\{-\gamma\big([0,t]_<^2\big)\Big\}\ge(1+\epsilon)t_n\log\log\log t_n\Big\}<\infty.$$

A standard procedure via Borel-Cantelli lemma gives the upper bound

$$\limsup_{t\to\infty}\frac{1}{t\log\log\log t}\Big(-\gamma\big([0,t]_<^2\big)\Big)\le\frac{1}{2\pi}\qquad a.s. \tag{4.4.5}$$

We now work on the lower bound. Recall that $t_n=\theta^n$. This time $\theta>1$ stands for a large number. For any $\lambda_1<(2\pi)^4$. By Theorem 4.3.1 an by scaling

$$\begin{aligned}
&\sum_n\mathbb{P}\Big\{\gamma\big([t_{n-1},t_n]_<^2\big)\ge\lambda_1 t_n\log\log t_n\Big\}\\
&=\sum_n\mathbb{P}\Big\{\gamma\big([0,1]_<^2\big)\ge\lambda_1\frac{t_n}{t_n-t_{n-1}}\log\log t_n\Big\}=\infty
\end{aligned}$$

for sufficiently large θ. Notice that

$$\gamma\big([t_{n-1},t_n]_<^2\big)\qquad n=1,2,\cdots$$

is an independent sequence. By Borel Cantelli lemma

$$\limsup_{n\to\infty}\frac{1}{t_n\log\log t_n}\gamma\big([t_{n-1},t_n]_<^2\big)\ge\lambda_1\qquad a.s. \tag{4.4.6}$$

In addition,

$$\begin{aligned}
\gamma\big([0,t_n]_<^2\big)&=\gamma\big([0,t_{n-1}]_<^2\big)+\gamma\big([t_{n-1},t_n]_<^2\big)\\
&\quad+\beta\big([0,t_{n-1}]\times[t_{n-1},t_n]\big)-\mathbb{E}\beta\big([0,t_{n-1}]\times[t_{n-1},t_n]\big)\\
&\ge\gamma\big([0,t_{n-1}]_<^2\big)+\gamma\big([t_{n-1},t_n]_<^2\big)+\mathbb{E}\beta\big([0,t_{n-1}]\times[t_{n-1},t_n]\big)
\end{aligned}$$

and the last term on the right hand side is equal to

$$\frac{1}{2\pi}\Big\{t_n\log t_n - t_{n-1}\log t_{n-1} - (t_n-t_{n-1})\log(t_n-t_{n-1})\Big\} = O(t_n).$$

By (4.4.5),

$$\liminf_{n\to\infty}\frac{1}{n\log\log n}\gamma\big([0,t_{n-1}]_<^2\big) = 0 \quad a.s.$$

Hence, it follows from (4.4.6) that

$$\limsup_{t\to\infty}\frac{1}{t\log\log t}\gamma\big([0,t]_<^2\big) \ge \kappa(2,2)^4 \quad a.s. \tag{4.4.7}$$

Write $u_n = \exp\{n^2\}$ $(n = 1,2,\cdots)$. By Theorem 4.3.2, for any $\lambda_2 < (2\pi)^{-1}$,

$$\begin{aligned}&\sum_n \mathbb{P}\Big\{-\gamma\big([u_{n-1},u_n]_<^2\big) \ge \lambda_2 u_n\log\log\log u_n\Big\}\\ &=\sum_n \mathbb{P}\Big\{-\gamma\big([0,1]_<^2\big) \ge \lambda_2\frac{u_n}{u_n-u_{n-1}}\log\log\log u_n\Big\} = \infty.\end{aligned}$$

By Borel-Cantelli lemma,

$$\limsup_{n\to\infty}\frac{1}{u_n\log\log\log u_n}\Big(-\gamma\big([u_{n-1},u_n]_<^2\big)\Big) \ge \frac{1}{2\pi} \quad a.s. \tag{4.4.8}$$

Notice that

$$-\gamma\big([0,u_n]_<^2\big) \ge -\gamma\big([0,u_{n-1}]_<^2\big) - \beta\big([0,u_{n-1}]\times[u_{n-1},u_n]\big) - \gamma\big([u_{n-1},u_n]_<^2\big).$$

To control the first two terms on the right hand side, we use the fact that as $n\to\infty$

$$u_n\log\log\log u_n \gg \sqrt{u_nu_{n-1}}\log\log u_n \gg u_{n-1}\log\log u_{n-1}.$$

Indeed, by (4.4.4) and (4.4.5),

$$\lim_{n\to\infty}\frac{1}{u_n\log\log\log u_n}\gamma\big([0,u_{n-1}]_<^2\big) = 0 \quad a.s.$$

According to Proposition 2.3.4,

$$\beta\big([0,u_{n-1}]\times[u_{n-1},u_n]\big) \overset{d}{=} \alpha\big([0,u_{n-1}]\times[0,u_n-u_{n-1}]\big).$$

For any integer $m \geq 1$, by (2.2.11),

$$\begin{aligned}
&\mathbb{E}\Big[\beta\big([0,u_{n-1}]\times[u_{n-1},u_n]\big)^m\Big]\\
&\leq \Big\{\mathbb{E}\Big[\alpha\big([0,u_{n-1}]^2\big)^m\Big]\Big\}^{1/2}\Big\{\mathbb{E}\Big[\alpha\big([0,u_n-u_{n-1}]^2\big)^m\Big]\Big\}^{1/2}\\
&= \big(u_{n-1}(u_n-u_{n-1})\big)^{m/2}\mathbb{E}\Big[\alpha\big([0,1]^2\big)^m\Big].
\end{aligned}$$

By Theorem 2.2.9 (with $d = p = 2$), therefore, there is a $c > 0$ such that

$$\sup_n \mathbb{E}\exp\Big\{c(u_{n-1}u_n)^{-1/2}\beta\big([0,u_{n-1}]\times[u_{n-1},u_n]\big)\Big\} < \infty.$$

By a standard way of using Chebyshev inequality one can see that for any $\epsilon > 0$,

$$\sum_n \mathbb{P}\Big\{\beta\big([0,u_{n-1}]\times[u_{n-1},u_n]\big) \geq \epsilon u_n \log\log\log u_n\Big\} < \infty.$$

By Borel-Cantelli lemma

$$\lim_{n\to\infty}\frac{1}{u_n\log\log\log u_n}\beta\big([0,u_{n-1}]\times[u_{n-1},u_n]\big) = 0 \qquad a.s.$$

Therefore, it follows from (4.4.8) that

$$\limsup_{t\to\infty}\frac{1}{t\log\log\log t}\Big(-\gamma\big([0,t]_<^2\big)\Big) \geq \frac{1}{2\pi} \qquad a.s. \tag{4.4.9}$$

Finally, Theorem 4.4.2 follows from (4.4.4), (4.4.5), (4.4.7) and (4.4.9). □

4.5 Notes and comments

Section 4.1

The Feynman-Kac formula goes back to Kac [98]. It has become a popular tool in the study on Markov processes and exists in literature in several different forms. We refer to the books by Fukushima ([77]), Rogers and Williams ([136], [137]), Sznitman (Chapter 1, [150]) for Feynman-Kac formula in the general context. Essentially, the well-known Donsker-Varadhan large deviations ([53], [55], [58]) for empirical measures of Markov processes are closely related to Feynman-Kac formula. Some idea in the proof of Theorem

4.1.6 comes from Remillard [132]. We also mention the book by Pinsky [131] for the applications of Feynman-Kac formula to the logarithmic moment generating functions of the occupation times of the diffusion processes.

Section 4.2

Theorem 4.2.1 is philosophically linked to the famous *Donsker-Varadhan's large deviations* for the empirical measure

$$\mathcal{L}_t(A) = \frac{1}{t}\int_0^t 1_{\{W(s)\in A\}}ds \qquad A \in \mathcal{B}(\mathbb{R})$$

Viewing $\mathcal{L}_t(\cdot)$ as a stochastic process taking values in $\mathcal{P}(\mathbb{R})$, the space of the probability measures on $\mathbb{R}$ equipped with the topology of weak convergence, Donsker and Varadhan ([52]) establish a weak version of the large deviation principle claiming

$$\lim_{t\to\infty}\frac{1}{t}\log \mathbb{P}\big\{\mathcal{L}_t(\cdot)\in\mathcal{U}\big\} = -\inf_{\mu\in\mathcal{U}} I(\mu)$$

for a class of $\mathcal{U}\subset\mathcal{P}(\mathbb{R})$, where the rate function

$$I(\mu) = \frac{1}{8}\int_{-\infty}^{\infty}\frac{|f'(x)|^2}{f(x)}dx$$

for absolute continuous probability measure μ with the density f.

Define the function Ψ on (a subset of) $\mathcal{P}(\mathbb{R})$ by

$$\Psi(\mu) = \bigg(\int_{-\infty}^{\infty} f^p(x)dx\bigg)^{1/p}$$

Varadhan's integral lemma (a functional version of Theorem 1.1.6, see Theorem 4.3.1 in [44] for full detail) morally supports (4.2.2) in the following way

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\bigg(\int_{-\infty}^{\infty}L^p(t,x)dx\bigg)^{1/p}\bigg\}\\ &=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\Big\{t\Psi\big(\mathcal{L}_t\big)\Big\} = \sup_{\mu\in\mathcal{P}(\mathbb{R}^d)}\Big\{\Psi(\mu)-I(\mu)\Big\}\\ &=\sup_f\bigg\{\bigg(\int_{-\infty}^{\infty}f^p(x)dx\bigg)^{1/p}-\frac{1}{8}\int_{-\infty}^{\infty}\frac{|f'(x)|^2}{f(x)}dx\bigg\}\end{aligned}$$

where the supremum on the left-hand side is taking for the probability density functions f on $\mathbb{R}$. Hence, it comes to (4.2.2) under the substitution $f(x) = g^2(x)$.

To make above heuristics into a mathematical argument, one has to overcome two major obstacles: the singularity of the function $\Psi(\mu)$ and the absence of the full large deviation for $\mathcal{L}_t$. Theorem 4.2.1 in the special case $p = 2$ was first obtained Mansmann [127] in this approach. To make it work, some very delicate approximation is adopted.

Prior to Mansmann's work, Donsker and Varadhan ([52], [53], [54], [55], [56] [57], [58]) had investigated the large deviations for their powerful results to a variety of the models appear to be the functionals of the empirical measure $\mathcal{L}_t$. In addition, see Bolthausen ([15]), ven den Berg, Bolthausen and den Hollander ([10], [11]) for the application of this approach to some models related to the topics of this book.

By excursion theory, Csáki, König and Shi ([39]) established the large deviations for self-intersection local times of Brownian bridges that take a form exactly same as Theorem 4.2.1. The result given in Theorem 4.2.1 was obtained by Chen and Li [21] in a more general form and the argument used here essentially comes from that paper.

The following exercise proposes an alternative approach (given in [21] and [24]) of establishing Theorem 3.3.2. We focus on the case $d = 1$ for simplicity.

Exercise 1. The crucial fact in this method is the inequality

$$\left(\int_{-\infty}^{\infty}\prod_{j=1}^p f_j(x)dx\right)^{1/p} \le \frac{1}{p}\sum_{j=1}^p\left(\int_{-\infty}^{\infty} f_j^p(x)dx\right)^{1/p}$$

where $f_1, \cdots, f_p \in \mathcal{L}^p(\mathbb{R})$ are non-negative functions.

(1). Recall that

$$\alpha\big([0,t]^p\big) = \int_{-\infty}^{\infty}\prod_{j=1}^p L_j(t,x)dx$$

as $d = 1$. Based on the above inequality and (4.2.2), establish a upper bound for the limit

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\Big(\alpha\big([0,t]^p\big)\Big)^{1/p}\right\}$$

where $\theta > 0$. Transform it into a upper bound for

$$\limsup_{t\to\infty}\frac{1}{t}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}\Big\{\mathbb{E}\alpha^m\big([0,t]^p\big)\Big\}^{1/p}$$

(2). Establish a lower bound for

$$\liminf_{t\to\infty}\frac{1}{t}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}\Big\{\mathbb{E}\alpha^m\big([0,t]^p\big)\Big\}^{1/p}$$

Remember, you are not allowed to use the lower bound established in Chapter 3.

Hint: Use Jensen inequality and Theorem 4.1.6.

(3). Combine above results to establish the large deviation for $\alpha\big([0,1]^p\big)$.

Duplantier-1

Section 4.3

Duplantier ([62]) points out that in quantum field theory, "Families of universal critical exponents are associated with intersection properties of sets of random walks". The large deviation in Theorem 4.3.1 and the related exponential integrability problem in (4.3.3) correspond to the continuum limit of the physical model known as Edwards model. We refer the reader to the papers by Westwater ([158], [159], [160]) and by Duplanier ([61], ([62])) for details. The result stated in Theorem 4.3.1 comes from the paper by Bass and Chen ([4]). Prior to [4], the critical exponential integrability given in (4.3.3) was studied by Yor ([167]) and by Le Gall ([123]) without identifying the constant λ_0. In the context of the random walks with discrete time and state space, same type of exponential integrability was obtained by Brydges and Slade ([17]). Further, Brydges and Slade provided a heuristic argument (p. 294, [17]) by using Donsker-Varadhan large deviation suggesting the relevance of Gagliardo-Nirenberg constant $\kappa(2,2)$. For the random walks of lattice values but continuous time, Bolthausen ([15]) obtained a large deviation with a form essentially same as Theorem 4.3.1, except that in his model, the self-intersection local time does not have to be renormalized.

Among all the works prior to ([4]), a private letter from B. Duplatier to M. Yor captures the most of the idea about the critical value λ_0 given in (4.3.3) (I Thank Dr. Yor and Dr. Duplantier for forwarding me the related document). In that letter, Duplatier pointed out (without proof) that

$$\lambda_0 = \frac{\pi}{2}\int_0^\infty rg^4(r)dr$$

where the function g solves the differential equation

$$g''(r) + \frac{1}{r}g'(r) = g(r) - g^3(r).$$

His observation is accurate. Indeed, let $f_0(x)$ be the positive, infinitely smooth, spherically symmetric ground state solution of the nolinear partial differential equation (3.4.1). By spherical symmetry $f(x) = g(|x|)$ for some function g on $\mathbb{R}^+$. One can see that when $d = p = 2$, g satisfies the ordinary differential equation proposed by Duplatier and that by (3.4.2)

$$\lambda_0 = \kappa(2,2)^{-4} = \frac{1}{4}\int_{\mathbb{R}^2} f^4(x)dx = \frac{\pi}{2}\int_0^\infty rg^4(r)dr.$$

In addition, from (3.4.3)

$$\lambda_0 \approx \pi \times 1.86225... \approx 5.85043...$$

This is very close to the numerical value provided by B. Duplantier.

A striking feature of the renormalized self-intersection local time is its asymptotically asymmetric tail, which was first observed by Le Gall ([123]). Theorem 4.3.2 sharpens Le Gall's observation. This result was obtained in Bass and Chen [4]. Finally, we mention a non-trivial extension of Theorem 4.3.1 and Theorem 4.3.2 to the setting of stable processes ([5]), where the stable counter part of Theorem 4.3.2 takes a drastically different form.

Section 4.4

The application of large deviations to the laws of the iterated logarithm becomes nearly standard: The large deviations provide the estimates needed for Borel-Cantelli-type of argument. The results in this section are expected consequences of the large deviations established in the previous sections.

Chapter 5

Intersections on lattices: weak convergence

-

Recall that a Z^d-valued random walk $\{S(n)\}$ is defined by

$$S(n) = X_1 + \cdots + X_n \quad n = 1, 2, \cdots$$

where $\{X_n\}$ is an i.i.d. Z^d-valued random variables. Unless claim other wise, we define $S(0) = 0$. Throughout we assume that X_1 is symmetric and has finite second moment and write Γ for the covariance matrix of X_1. We assume that the smallest group that supports these random walks is $\mathbb{Z}^d$.

Let $S_1(n), \cdots, S_p(n)$ be independent copies of $S(n)$. For any $\Delta \subset \mathbb{Z}^+$, set

$$S_j(\Delta) = \{S_j(k); \quad k \in \Delta\}.$$

Here is a list of the models of interest

1. *Self-intersection local time*

$$Q_n^{(p)} = \sum_{1 \le k_1 < \cdots < k_p \le n} 1_{\{S(k_1) = \cdots = S(k_p)\}}.$$

An important special case is when $p = 2$. In this case we write $Q_n = Q_n^{(2)}$.

2. *Range of the random walk*

$$R_n = \#\big\{S[1, n]\big\} = \#\big\{S(1), \cdots, S(n)\big\}.$$

3. *Mutual intersection local time*

$$I_n = \sum_{k_1,\cdots,k_p=1}^{n} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}}.$$

4. *Intersection of the independent ranges*

$$J_n = \#\big\{S_1[1,n]\cap\cdots\cap S_p[1,n]\big\}.$$

The goal of this chapter is to establish weak laws of convergence for these models. The limiting distributions of our theorems are these of Brownian intersection local times.

5.1 Preliminary on random walks

Let

$$\varphi(\lambda) = \sum_{x\in\mathbb{Z}^d} e^{i\lambda\cdot x}\mathbb{P}\{X_1 = x\}$$

be the characteristic function of X_1. Then $\varphi(\lambda)$ is a real, periodic function with period 2π:

$$\varphi(\lambda + 2\pi x) = \varphi(\lambda) \qquad x \in \mathbb{Z}^d.$$

We now discuss another (unrelated) kind of (a)periodicity. The *period* of the random walk $\{S(n)\}$ is defined as the greatest common factor of the set

$$\big\{n \ge 1;\ \ \mathbb{P}\{S_n = 0\} > 0\big\}.$$

$\{S(n)\}$ is said to be *aperiodic* if its period is 1. It is known that $\{S(n)\}$ is aperiodic if and only if

$$|\varphi(\lambda)| < 1 \qquad \forall\lambda \in [-\pi,\pi]^d \setminus \{0\}.$$

The aperiodicity is a good property under which things usually become easier. On the other hand, the aperiodicity is not satisfied by even in some important special case. A random walk on $\mathbb{Z}^d$ is called *simple random walk* if the distribution of its increment is given by

$$\mathbb{P}\{X_1 = \pm e_j\} = \frac{1}{2d} \qquad j = 1,\cdots d$$

where $\{e_1,\cdots e_d\} \subset \mathbb{Z}^d$ is the canonical orthogonal basis in $\mathbb{Z}^d$. It is easy to see that the period of a simple random walk is 2.

Lemma 5.1.1 . *Assume that $\{S(n)\}$ is aperiodic.*

(1). There is $\delta > 0$ such that

$$|\varphi(\lambda)| \le \exp\Big\{ -\delta|\lambda|^2\Big\} \qquad \lambda \in [-\pi, \pi]^d. \tag{5.1.1}$$

(2).

$$\sup_{x\in\mathbb{Z}^d} \Big| n^{d/2}\mathbb{P}\{S(n) = x\} - \frac{1}{(2\pi)^{d/2}\det(\Gamma)^{1/2}} \exp\Big\{ -\frac{\langle x, \Gamma^{-1}x\rangle}{2n}\Big\}\Big| \tag{5.1.2}$$
$$\longrightarrow 0 \qquad (n \to \infty)$$

In particular,

$$\mathbb{P}\{S_n = 0\} \sim \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Gamma)}} n^{-d/2} \qquad (n \to \infty). \tag{5.1.3}$$

(3). There is a constant $C > 0$ such that

$$\Big|\mathbb{P}\{S(n) = x\} - \mathbb{P}\{S(n) = y\}\Big| \le Cn^{-d/2}\frac{|x-y|}{\sqrt{n}} \tag{5.1.4}$$

for any $x, y \in \mathbb{Z}^d$ and any $n = 1, 2, \cdots$.

Proof. . To prove the first part, notice that by Taylor expansion

$$0 \le \varphi(\lambda) = \mathbb{E}e^{i\lambda\cdot X_1} \sim 1 - \frac{1}{2}\lambda\cdot\Gamma\lambda \qquad (|\lambda| \to 0).$$

Notice that there is a $c > 0$ such that $\lambda\cdot\Gamma\lambda \ge c|\lambda|^2$. This concludes that there are $\epsilon > 0$ and $c_1 > 0$ such that

$$|\varphi(\lambda)| \le \exp\Big\{ -c_1|\lambda|^2\Big\} \qquad |\lambda| \le \epsilon.$$

Write $T_\epsilon = \{\lambda \in [-\pi, \pi]^d; \quad |\lambda| \ge \epsilon\}$. By aperiodicity we have that

$$\theta \equiv \sup_{\lambda\in T_\epsilon} |\varphi(\lambda)| < 1.$$

Consequently, there is $c_2 > 0$ such that

$$\theta \le \exp\Big\{ -c_2 \sup_{\lambda\in[-\pi,\pi]^d} |\lambda|^2\Big\}.$$

Letting $\delta = \min\{c_1, c_2\}$ give the conclusion in the first part.

The proof of (5.1.2), (5.1.3) and (5.1.4) relies on (5.1.1) and Fourier transformation. For similarity, we only prove (5.1.3). By Fourier inversion ((B.4), Appendix),

$$\mathbb{P}\{S_n = x\} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-i\lambda\cdot x}\mathbb{E}e^{i\lambda\cdot S_n}d\lambda$$
$$= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-i\lambda\cdot x}\varphi(\lambda)^n d\lambda.$$

Take $x = 0$. By variable substitution

$$n^{d/2}\mathbb{P}\{S_n = 0\} = \frac{1}{(2\pi)^d}\int_{[-\sqrt{n}\pi,\sqrt{n}\pi]^d}\varphi(\lambda/\sqrt{n})^n d\lambda.$$

By the classic central limit theorem

$$\lim_{n\to\infty}\varphi(\lambda/\sqrt{n})^n = \exp\Big\{-\frac{1}{2}\lambda\cdot\Gamma\lambda\Big\}$$

and by (5.1.1),

$$1_{[-\sqrt{n}\pi,\sqrt{n}\pi]^d}(\lambda)\varphi(\lambda/\sqrt{n})^n \le \exp\big\{-\delta|\lambda|^2\big\}.$$

Applying dominated convergence theorem gives

$$\lim_{n\to\infty} n^{d/2}\mathbb{P}\{S_n = 0\} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\Big\{-\frac{1}{2}\langle\lambda,\Gamma\lambda\rangle\Big\}d\lambda = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Gamma)}}.$$

□

It is easy to see that the aperiodicity is necessary in Lemma 5.1.1. In some other situations, however, the result is first proved under aperiodicity and then is extended to the general case by some method. Define, for example,

$$g_n = \sum_{k=0}^n \mathbb{P}\{S(n) = 0\} \qquad n = 0, 1, 2, \cdots. \tag{5.1.5}$$

We call g_n the *partial green's function* of the random walk.

Lemma 5.1.2 *When $n\to\infty$,*

$$g_n \sim \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma}\sqrt{n} & d = 1 \\[2ex] \dfrac{1}{2\pi\sqrt{\det(\Gamma)}}\log n & d = 2 \\[2ex] O(1) & d \ge 3. \end{cases} \tag{5.1.6}$$

Proof. With aperiodicity, this lemma holds immediately from (5.1.3). We now extend it to the general case. For similarity we only consider the case $d = 1$ in the following argument.

The approach we shall adopt is called *resolvent approximation* . Let $0 < \theta < 1$ be fixed and let $\{\delta_n\}$ be a Bernoulli sequence independent of $\{S(n)\}$ such that

$$\mathbb{P}\{\delta_n = 1\} = 1 - \mathbb{P}\{\delta_n = 0\} = \theta.$$

Write $\tau_0 = 0$ and

$$\tau_n = \min\{k > \tau_{n-1};\quad \delta_k = 1\} \qquad n = 1, 2, \cdots .$$

Then $\{\tau_n - \tau_{n-1}\}_{n\geq 1}$ is an i.i.d. sequence with common (geometric) distribution

$$\mathbb{P}\{\tau_1 = k\} = \theta(1-\theta)^{k-1} \qquad k = 1, 2, \cdots .$$

Notice that $\{S(\tau_n)\}$ is a random walk whose increment has the distribution

$$\mathbb{P}\{S(\tau_1) = x\} = \sum_{k=1}^{\infty} \theta(1-\theta)^{k-1}\mathbb{P}\{S(k) = x\}.$$

In particular,

$$\mathrm{Var}\big(S(\tau_1)\big) = \mathbb{E}\tau_1 \cdot \sigma^2 = \theta^{-1}\sigma^2$$

and, $\mathbb{P}\{S(\tau_1) = 0\} > 0$ implies that $\{S(\tau_n)\}$ is aperiodic. $\{S(\tau_n)\}$ is called *resolvent random walk* of $\{S(n)\}$ in literature.

Apply what we know under aperiodicity to $\{S(\tau_n)\}$,

$$\sum_{k=1}^{n} \mathbb{P}\{S(\tau_k) = 0\} \sim \frac{\sqrt{\theta}}{\sqrt{2\pi}\sigma}\sqrt{n} \qquad (n \to \infty).$$

By definition one can see that

$$\sum_{k=1}^{n} 1_{\{S(\tau_k)=0\}} = \sum_{k=1}^{\tau_n} \delta_k 1_{\{S(k)=0\}} = \sum_{k=1}^{\tau_n} \theta 1_{\{S(k)=0\}} + \sum_{k=1}^{\tau_n} (\delta_k - \theta) 1_{\{S(k)=0\}}.$$

By the classic law of large numbers, $\tau_n/n \to \theta^{-1}$ in several ways (in moment, in probability, with probability, etc.). In particular, one can see that the expectation of the second term on the right hand side is of the order $o(\sqrt{n})$ and is therefore negligible. As for the first term,

$$\mathbb{E}\sum_{k=1}^{\tau_n} \theta 1_{\{S(k)=0\}} = \mathbb{E}g_{\tau_n} \sim g_{[n\theta^{-1}]} \qquad (n \to \infty).$$

Summarizing our argument,

$$g_{[n\theta^{-1}]} \sim \frac{\sqrt{\theta}}{\sqrt{2\pi}\sigma}\sqrt{n} \qquad (n \to \infty)$$

which leads to desired conclusion. □

Write

$$T_x = \inf\{k \geq 1; \quad S(k) = x\} \qquad x \in \mathbb{Z}^d. \tag{5.1.7}$$

Then for each x, T_x is a stopping time and is called *hitting time* in literature.

Lemma 5.1.3 *(The first entry formula). For any $x \in \mathbb{Z}^d$,*

$$\mathbb{P}\{S(k) = x\} = \sum_{j=1}^{k} \mathbb{P}\{T_x = j\}\mathbb{P}\{S(k-j) = 0\}. \tag{5.1.8}$$

Consequently,

$$\sum_{k=1}^{n} \mathbb{P}\{S(k) = x\} = \sum_{k=1}^{n} \mathbb{P}\{T_x = k\} g_{n-k} \tag{5.1.9}$$

where $g_0 = 0$ and $\mathbb{P}\{S(0) = 0\} = 1$.

Proof. The first identity follows from the following computation:

$$\mathbb{P}\{S(k) = x\} = \mathbb{P}\{S(k) = x,\ T_x \leq k\} = \sum_{j=1}^{k} \mathbb{P}\{S(k) = x,\ T_x = j\}$$

$$= \sum_{j=1}^{k} \mathbb{P}\{S(k) - S(j) = 0,\ T_x = j\} = \sum_{j=1}^{k} \mathbb{P}\{T_x = j\}\mathbb{P}\{S(k-j) = 0\}.$$

Summing up on the both side of (5.1.8) gives (5.1.9). □

Theorem 5.1.4

$$\mathbb{P}\{T_0 < \infty\} = \begin{cases} 1 & d = 1, 2 \\[2ex] \dfrac{\displaystyle\sum_{k=1}^{\infty} \mathbb{P}\{S(k) = 0\}}{1 + \displaystyle\sum_{k=1}^{\infty} \mathbb{P}\{S(k) = 0\}} & d \geq 3. \end{cases} \tag{5.1.10}$$

In particular,

$$\mathbb{P}\{S(k) = 0 \text{ infinitely often}\} = \begin{cases} 1 & d \le 2 \\ 0 & d \ge 3. \end{cases} \tag{5.1.11}$$

Proof. *Proof of (5.1.10).* Due to similarity, we only consider the case $d \ge 3$. Taking $x = 0$ in (5.1.8) and summing up the both side

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbb{P}\{S(k) = 0\} &= \sum_{k=1}^{\infty}\sum_{j=1}^{k} \mathbb{P}\{T_0 = j\}\mathbb{P}\{S(k-j) = 0\} \\ &= \sum_{j=1}^{\infty} \mathbb{P}\{T_0 = j\} \sum_{k=j}^{\infty} \mathbb{P}\{S(k-j) = 0\} \\ &= \mathbb{P}\{T_0 < \infty\}\Big\{1 + \sum_{k=1}^{\infty} \mathbb{P}\{S(k) = 0\}\Big\}.\end{aligned}$$

Proof of (5.1.11). As $d \ge 3$, (5.1.11) follows from Borel-Cantelli lemma and the fact

$$\sum_{k=1}^{\infty} \mathbb{P}\{S(k) = 0\} < \infty.$$

Let $d \le 2$. For each $m \ge 1$, let $T_0(m)$ be the the time when $\{S(k)\}$ is having mth hit of 0. We take $T_0(m) = \infty$ if the mth hit never come. Under our notation, $T_0(1) = T_0$. Clearly

$$\mathbb{P}\Big\{\sum_{k=1}^{\infty} 1_{\{S(k)=0\}} \ge m\Big\} = \mathbb{P}\big\{T_0(m) < \infty\big\}.$$

By the conclusion from the first part

$$\begin{aligned}\mathbb{P}\{T_0(m) < \infty\} &= \mathbb{P}\big\{T_0(m-1) < \infty,\ T_0(m) < \infty\big\} \\ &= \sum_{1 \le j < k < \infty} \mathbb{P}\big\{T_0(m-1) = j,\ T_0(m) = k\big\} \\ &= \sum_{1 \le j < k < \infty} \mathbb{P}\{T_0(m-1) = j\}\mathbb{P}\{T_0 = k - j\} \\ &= \mathbb{P}\{T_0 < \infty\} \sum_{j=1}^{\infty} \mathbb{P}\{T_0(m-1) = j\} \\ &= \mathbb{P}\{T_0(m-1) < \infty\}.\end{aligned}$$

Repeating the procedure gives

$$\mathbb{P}\Big\{\sum_{k=1}^{\infty} 1_{\{S(k)=0\}} \geq m\Big\} = \mathbb{P}\{T_0(m) < \infty\} = \mathbb{P}\{T_0 < 1\} = 1$$

for $m = 1, 2, \cdots$. Consequently,

$$\mathbb{P}\Big\{\sum_{k=1}^{\infty} 1_{\{S(k)=0\}} = \infty\Big\} = \lim_{m\to\infty} \mathbb{P}\Big\{\sum_{k=1}^{\infty} 1_{\{S(k)=0\}} \geq m\Big\} = 1.$$

$\square$

Theorem 5.1.4 shows that the random walk is *recurrent* when $d = 1, 2$ and *transient* when $d \geq 3$. A quantity we use very often later is *the probability of no returning* γ_S defined as

$$\gamma_S = \mathbb{P}\big\{S(k) \neq 0 \text{ for all } n \geq 1\big\}$$

Another way to write (5.1.10) is

$$\gamma_S = \begin{cases} 0 & d = 1, 2 \\ \Big(\displaystyle\sum_{k=0}^{\infty} \mathbb{P}\{S(k) = 0\}\Big)^{-1} & d \geq 3. \end{cases} \tag{5.1.12}$$

5.2 Intersection in 1-dimension

The local time of the random walk is given as

$$l(n, x) = \sum_{k=1}^{n} 1_{\{S(k)=x\}} \qquad x \in \mathbb{Z}^d. \tag{5.2.1}$$

In the rest of the section we let $d = 1$ and $p \geq 2$. By a combinatorial argument,

$$\sum_{x\in\mathbb{Z}} l^p(n, x) = \sum_{k_1,\cdots,k_p=1}^{n} 1_{\{S(k_1)=\cdots=S(k_p)\}} \tag{5.2.2}$$

$$= n + \sum_{j=2}^{p} \binom{p-1}{j-1} j! \sum_{1\leq k_1<\cdots<k_j\leq n} 1_{\{S(k_1)=\cdots=S(k_j)\}}.$$

We shall see that the typical order of the left hand side is $n^{\frac{p+1}{2}}$ as $n \to \infty$. Consequently,

$$\sum_{x\in\mathbb{Z}} l^p(n,x) - p! \sum_{1\le k_1<\cdots<k_p\le n} 1_{\{S(k_1)=\cdots=S(k_p)\}} \\ = O\big(n^{p/2}\big) \qquad (n\to\infty) \tag{5.2.3}$$

in the sense of the weak convergence. Therefore, the weak law for the p-multiple self-intersection is reduced to the study of the weak law for the p-square

$$\sum_{x\in\mathbb{Z}} l^p(n,x).$$

Define

$$\zeta_n(x) = \begin{cases} n^{-1/2} l(n,k) & x = \dfrac{k}{\sqrt{n}} \text{ for all } k\in\mathbb{Z} \\ \text{linear} & \text{else} \end{cases}$$

or,

$$\zeta_n(x) = \frac{1}{\sqrt{n}} l(n,k) + \Big(x - \frac{k}{\sqrt{n}}\Big)\Big[l(n,k+1) - l(n,k)\Big]$$

for $x \in \Big[\dfrac{k}{\sqrt{n}}, \dfrac{k+1}{\sqrt{n}}\Big]$ and for all $k\in\mathbb{Z}$.

Recall that for any $t>0$, the local time $L(t,x)$ of an 1-dimensional Brownian motion $W(t)$ is defined as the density of the occupation measure

$$\mu_t(A) = \int_0^t 1_B\big(W(s)\big)ds.$$

Here we view $\zeta_n(\cdot)$ and $L(t,\cdot)$ random variables taking values in the Banach space $\mathcal{L}^p(\mathbb{R}^d)$.

Theorem 5.2.1 *When $n\to\infty$,*

$$\zeta_n(\cdot) \xrightarrow{d} \sigma^{-1} L\big(1, \sigma^{-1}(\cdot)\big) \tag{5.2.4}$$

where $\sigma^2>0$ is the variance of the random walk $\{S(n)\}$.

The hard part of the proof is to show that the sequence $\{\zeta_n\}$ is uniformly tight. For this we need the following moment estimate.

Lemma 5.2.2 *Assume that the random walk $S(n)$ is aperiodic. There is a $C > 0$ such that*

$$\mathbb{E}\Big|l(n,x) - l(n,y)\Big|^{2l} \le (2l)! C^l n^{\frac{3l}{4}} |x-y|^{l/2} \tag{5.2.5}$$

for any integers $l, n \ge 1$ and for any $x, y \in \mathbb{Z}$.

Let $\zeta_n(\cdot)$ be defined as in Theorem 5.2.1. Then

$$\mathbb{E}\Big|\zeta_n(x) - \zeta_n(y)\Big|^{2l} \le (2l)! C^l |x-y|^{l/2} \tag{5.2.6}$$

for any integers $l, n \ge 1$ and for any $x, y \in \mathbb{R}$.

Proof. We first establish the bound

$$\sum_{k=1}^{n} |\mathbb{P}\{S(k) = x\} - \mathbb{P}\{S(k) = 0\}| \le C n^{1/4} |x|^{1/2} \tag{5.2.7}$$

where $C > 0$ represents a constant independent of n and $x \in \mathbb{Z}$ (To simplify the notation, we allow C to be different from place to place in our proof). Indeed, by Lemma 5.1.1,

$$\begin{aligned} &\sum_{k=1}^{[\sqrt{n}]|x-y|} |\mathbb{P}\{S(k) = x\} - \mathbb{P}\{S(k) = y\}| \\ &\le C \sum_{k=1}^{[\sqrt{n}]|x-y|} k^{-1/2} = C n^{1/4} |x-y|^{1/2}. \end{aligned}$$

By (5.1.4),

$$\begin{aligned} &\sum_{k=[\sqrt{n}]|x-y|+1}^{n} |\mathbb{P}\{S(k) = x\} - \mathbb{P}\{S(k) = y\}| \\ &\le C \frac{|x-y|}{n^{1/4}\sqrt{|x-y|}} \sum_{k=1}^{n} k^{-1/2} \le C n^{1/4} |x-y|^{1/2}. \end{aligned}$$

Write $\eta(z) = 1_x(z) - 1_y(z)$. Then

$$\mathbb{E}\big[l(n,x) - l(n,y)\big]^m = \mathbb{E}\Big[\sum_{k=1}^{n} \eta\big(S(k)\big)\Big]^m = \sum_{l_1,\cdots,l_m=1}^{n} \mathbb{E} \prod_{k=1}^{m} \eta\big(S(l_k)\big).$$

Break the summation on the right hand side into $m!$ parts. In each part, the index $l_1, \cdots l_m$ is in non-decreasing order. For fixed $l_1 \le \cdots \le l_m$,

$$
\begin{aligned}
&\mathbb{E}\prod_{k=1}^{m}\eta\big(S(l_k)\big)\\
&=\mathbb{E}\Big\{\Big(\prod_{k=1}^{m-2}\eta\big(S_(l_k)\big)\Big)1_x\big((S_(l_{m-1})\big)\eta\big(S(l_m)-S(l_{m-1})+x\big)\Big\}\\
&-\mathbb{E}\Big\{\Big(\prod_{k=1}^{m-2}\eta\big(S(l_k)\big)\Big)1_y\big((S_(l_{m-1})\big)\eta\big(S(l_m)-S(l_{m-1})+y\big)\Big\}.
\end{aligned}
$$

By Markov property, the right hand side is equal to

$$
\begin{aligned}
&\mathbb{E}\Big\{\Big(\prod_{k=1}^{m-2}\eta\big(S(l_k)\big)\Big)1_x\big((S(l_{m-1})\big)\Big\}\mathbb{E}\eta\big(S(l_m-l_{m-1})+x\big)\\
&-\mathbb{E}\Big\{\Big(\prod_{k=1}^{m-2}\eta\big(S(l_k)\big)\Big)1_y\big((S(l_{m-1})\big)\Big\}\mathbb{E}\eta\big(S(l_m-l_{m-1})+y\big).
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&-\mathbb{E}\eta\big(S(l_m-l_{m-1})+y\big)=\mathbb{E}\eta\big(S(l_m-l_{m-1})+x\big)\\
&=\mathbb{P}\{S(l_m-l_{m-1})=0\}-\mathbb{P}\{S(l_m-l_{m-1})=x-y\}.
\end{aligned}
$$

Hence,

$$
\mathbb{E}\prod_{k=1}^{m}\eta\big(S(l_k)\big)=\mathbb{E}\eta\big(S(l_m-l_{m-1})+x\big)\mathbb{E}\Big\{\Big(\prod_{k=1}^{m-2}\eta\big(S(l_k)\big)\Big)\tilde{\eta}(S(l_{m-1})\Big\}
$$

where $\tilde{\eta}(z)=1_x(z)+1_y(z)$. Similarly,

$$
\begin{aligned}
&\mathbb{E}\Big\{\Big(\prod_{k=1}^{m-2}\eta\big(S(l_k)\big)\Big)\tilde{\eta}(S(l_{m-1})\Big\}\\
&=\mathbb{E}\Big\{\Big(\prod_{k=1}^{m-3}\eta\big(S(l_k)\big)\Big)1_x\big((S(l_{m-2})\big)\Big\}\mathbb{E}\tilde{\eta}\big(S(l_{m-1}-l_{m-2})+x\big)\\
&-\mathbb{E}\Big\{\Big(\prod_{k=1}^{m-3}\eta\big(S(l_k)\big)\Big)1_y\big((S(l_{m-2})\big)\Big\}\mathbb{E}\tilde{\eta}\big(S(l_{m-1}-l_{m-2})+y\big)\\
&=\mathbb{E}\tilde{\eta}\big(S(l_{m-1}-l_{m-2})+x\big)\mathbb{E}\Big(\prod_{k=1}^{m-2}\eta\big(S(l_k)\big)\Big)
\end{aligned}
$$

where the last step follows from the fact that

$$\begin{aligned}&\mathbb{E}\tilde{\eta}\big(S(l_{m-1}-l_{m-2})+x\big)=\mathbb{E}\tilde{\eta}\big(S(l_{m-1}-l_{m-2})+y\big)\\&=\mathbb{P}\{S(l_m-l_{m-1})=0\}+\mathbb{P}\{S(l_m-l_{m-1})=x-y\}.\end{aligned}$$

Summarizing our argument

$$\begin{aligned}&\mathbb{E}\prod_{k=1}^{m}\eta\big(S(l_k)\big)\\&=\mathbb{E}\tilde{\eta}\big(S(l_{m-1}-l_{m-2})+x\big)\mathbb{E}\eta\big(S(l_m-l_{m-1})+x\big)\mathbb{E}\Big(\prod_{k=1}^{m-2}\eta\big(S(l_k)\big)\Big).\end{aligned}$$

When $m=2l$ is even,

$$\mathbb{E}\prod_{k=1}^{2l}\eta\big(S(l_k)\big)=\prod_{k=1}^{l}\mathbb{E}\tilde{\eta}\big(S(l_{2k-1}-l_{2k-2})+x\big)\mathbb{E}\eta\big(S(l_{2k}-l_{2k-1})+x\big).$$

Therefore,

$$\begin{aligned}&\mathbb{E}\big[l(n,x)-l(n,y)\big]^{2l}\\&\le(2l)!\Big\{\sum_{k=1}^{n}\mathbb{E}\tilde{\eta}\big(S(k)+x\big)\Big\}^{l}\Big\{\sum_{k=1}^{n}\big|\mathbb{E}\eta\big(S(k)+x\big)\big|\Big\}^{l}\\&\le(2l)!\Big\{\sum_{k=1}^{n}\Big(\mathbb{P}\{S(k)=0\}+\mathbb{P}\{S(k)=x-y\}\Big)\Big\}^{l}\\&\times\Big\{\sum_{k=1}^{n}\big|\mathbb{P}\{S(k)=0\}-\mathbb{P}\{S(k)=x-y\}\big|\Big\}^{l}.\end{aligned}$$

Thus, (5.2.5) follows from Lemma 5.1.1 and (5.2.7).

By (5.2.5) we can easily see that for $|x-y|\ge 1/\sqrt{n}$,

$$\mathbb{E}\Big[\zeta_n(x)-\zeta_n(y)\Big]^{2l}\le(2l)!C^l n^{-l/4}|[\sqrt{n}x]-[\sqrt{n}y]|^{l/2}.$$

Thus, (5.2.6) follows from the simple fact that

$$|[\sqrt{n}x]-[\sqrt{n}y]|\le 3\sqrt{n}|x-y|.$$

Assume that $|x-y|<1/\sqrt{n}$. If x,y is in the same interval $\Big[\frac{k}{\sqrt{n}},\frac{k+1}{\sqrt{n}}\Big]$. Then we have

$$\zeta_n(x)-\zeta_n(y)=(y-x)\Big[l(n,k)-l(n,k+1)\Big].$$

By (5.2.5),

$$\mathbb{E}\Big[\zeta_n(x)-\zeta_n(y)\Big]^{2l} \le |y-x|^{2l}(2l)!C^l n^{3l/4} \le (2l)!C^l|y-x|^{l/2}.$$

If x, y is not in the same interval, we may assume that $x < y$. In this case, there must have k such that $x \in \Big[\frac{k-1}{\sqrt{n}}, \frac{k}{\sqrt{n}}\Big]$ and $y \in \Big[\frac{k}{\sqrt{n}}, \frac{k+1}{\sqrt{n}}\Big]$. By definition of ζ_n,

$$\begin{aligned}\zeta_n(y)-\zeta_n(x) &= \Big(y-\frac{k}{\sqrt{n}}\Big)\Big[l(n,k+1)-l(n,k)\Big]\\ &\quad+\Big(\frac{k}{\sqrt{n}}-x\Big)\Big[l(n,k)-l(n,k-1)\Big].\end{aligned}$$

Notice that

$$0 \le \Big(y-\frac{k}{\sqrt{n}}\Big),\ \Big(\frac{k}{\sqrt{n}}-x\Big). \le |x-y|$$

By (5.2.5),

$$\mathbb{E}\Big[\zeta_n(x)-\zeta_n(y)\Big]^{2l} \le |y-x|^{2l}(2l)!C^l n^{3l/4} \le (2l)!C^l|y-x|^{l/2}.$$

□

Proof of Theorem 5.2.1. By the method of resolvent approximation introduced in Section 5.1, we may assume that the random walk $\{S(n)\}$ is aperiodic. Let $q > 1$ be the conjugate of p and let $f \in \mathcal{L}^q(\mathbb{R})$ be continuous.

$$\begin{aligned}&\int_{\mathbb{R}} f(x)\zeta_n(x)\\ &= \sum_{k\in\mathbb{Z}}\int_{\frac{k}{\sqrt{n}}}^{\frac{k+1}{\sqrt{n}}} f(x)\Big\{\frac{1}{\sqrt{n}}l(n,k)+\Big(x-\frac{k}{\sqrt{n}}\Big)\Big[l(n,k+1)-l(n,k)\Big]\Big\}dx\\ &= \frac{1}{n}\sum_{k\in\mathbb{Z}}\int_k^{k+1} f\Big(\frac{x}{\sqrt{n}}\Big)\Big\{l(n,k)+\Big(\frac{x-k}{\sqrt{n}}\Big)\Big[l(n,k+1)-l(n,k)\Big]\Big\}dx\\ &= o(1)+\frac{1}{n}\sum_{k\in\mathbb{Z}} f\Big(\frac{k}{\sqrt{n}}\Big)l(n,k) = o(1)+\frac{1}{n}\sum_{l=1}^n f\Big(\frac{S(l)}{\sqrt{n}}\Big).\end{aligned}$$

By invariance principle

$$\begin{aligned}\int_{\mathbb{R}} f(x)\zeta_n(x) &\xrightarrow{d} \int_0^1 f\big(\sigma W(1)\big)dx\\ &= \int_{\mathbb{R}} f(\sigma x)L(1,x)dx = \sigma^{-1}\int_{\mathbb{R}} f(x)L(1,\sigma^{-1}x)dx.\end{aligned}$$

It remains to show that that $\zeta_n(\cdot)$ is uniformly tight in $\mathcal{L}^p(\mathbb{R})$. Given $\epsilon > 0$ there is a $a > 0$ such that

$$\mathbb{P}\Big\{\max_{k\le n}|S(k)| \ge a\sqrt{n}\Big\} < \epsilon/2.$$

Recall the classic fact that (Theorem 6.5, [23])

$$l(n,0)/\sqrt{n} \xrightarrow{d} \sigma^{-1}|U|$$

where U is a standard normal random variable. In particular, the real random sequence $\zeta_n(0) = l(n,0)/\sqrt{n}$ is a uniformly tight in $\mathbb{R}$. By (5.2.6) and Kolmogorov continuity theorem (Theorem D.7, Appendix), conditioning on the event $\Big\{\max_{k\le n}|S(k)| \le a\sqrt{n}\Big\}$, $\{\zeta_n(\cdot)\}$ is uniformly tight in $C[-a,a]$. Consequently there is a compact set $K \subset C[-a,a]$ such that

$$\mathbb{P}\Big\{\zeta_n(\cdot) \notin K,\ \max_{k\le n}|S(k)| \ge a\sqrt{n}\Big\} < \epsilon/2.$$

Viewing K as a subset of $C(\mathbb{R})$ such that every $f(\cdot) \in K$ is supported on $[-a,a]$, we have that K is compact in $C(\mathbb{R})$ and that

$$\mathbb{P}\Big\{\zeta_n(\cdot) \notin K\Big\} < \epsilon.$$

Notice that

$$\int_{\mathbb{R}} \zeta_n(x)dx = 1 \qquad n = 1, 2, \cdots .$$

Hence, $\mathbb{P}\{\zeta_n(\cdot) \notin A\} < \epsilon$, where

$$A \equiv K \cap \Big\{f \in C(\mathbb{R});\quad f \ge 0 \text{ and } \int_{\mathbb{R}} f(x)dx = 1\Big\}$$

is relatively compact in $\mathcal{L}^p(\mathbb{R})$.

Take $\widetilde{K}$ as the closure of A in $\mathcal{L}^p(\mathbb{R})$. Then $\widetilde{K}$ is compact in $\mathcal{L}^p(\mathbb{R})$ and $\mathbb{P}\{\zeta_n(\cdot) \notin \widetilde{K}\} < \epsilon$. $\square$

Here is how we apply Theorem 5.2.1: By a straightforward computation:

$$\begin{aligned}
&\int_{\mathbb{R}} \zeta_n^p(x)dx \\
&= \frac{1}{n^{\frac{p+1}{2}}} \sum_{k\in\mathbb{Z}} \int_k^{k+1} \Big\{l(n,k) + (x-k)\big[l(n,k+1) - l(n,k)\big]\Big\}dx \\
&= \frac{1}{n^{\frac{p+1}{2}}} \Big\{o(1) + \sum_{x\in\mathbb{Z}} l^p(n,x)\Big\}
\end{aligned}$$

where the second step follows from (5.2.5). By Theorem 5.2.1 on the other hand,

$$\int_{\mathbb{R}} \zeta_n^p(x)dx \xrightarrow{d} \sigma^{-p}\int_{\mathbb{R}} L^p(1,\sigma^{-1}x)dx = \sigma^{-(p-1)}\int_{\mathbb{R}} L^p(1,x)dx$$

Moreover, let $S_1(n),\cdots,S_p(n)$ be independent copies of $S(n)$ and consider the mutual intersection local time

$$I_n = \#\Big\{(k_1,\cdots,k_p)\in[1,n]^p;\quad S_1(k_1)=\cdots=S_p(k_p)\Big\}.$$

Then we have

$$I_n = \sum_{x\in\mathbb{Z}}\prod_{j=1}^p l_j(n,x) = n^{\frac{p+1}{2}}\Big\{o(1)+\int_{\mathbb{R}}\prod_{j=1}^p \zeta^{(j)}(x)dx\Big\}$$

where $l_j(n,x)$ is the local time of $S_j(n)$ and the $\mathcal{L}^p(\mathbb{R})$-valued random variable $\zeta_n^{(j)}(\cdot)$ is generated from $l_j(n,x)$ in the same way we generate $\zeta_n(\cdot)$ $(j=1,\cdots,p)$. By Theorem 5.2.1,

$$\Big(\zeta_n^{(1)}(\cdot),\cdots,\zeta_n^{(p)}(\cdot)\Big) \xrightarrow{d} \sigma^{-1}\Big(L_1\big(1,\sigma^{-1}(\cdot)\big),\cdots,L_1\big(1,\sigma^{-1}(\cdot)\big)\Big)$$

in the product space $\bigotimes_{j=1}^p \mathcal{L}^p(\mathbb{R})$, where $L_1(t,x),\cdots,L_p(t,x)$ are the local times of independent 1-dimensional Brownian motions $W_1(t),\cdots,W_p(t)$, respectively. This leads to

$$\begin{aligned}\int_{\mathbb{R}}\prod_{j=1}^p \zeta^{(j)}(x)dx &\xrightarrow{d} \sigma^{-p}\int_{\mathbb{R}}\prod_{j=1}^p L_j(1,\sigma^{-1}x)dx \\ &= \sigma^{-(p-1)}\int_{\mathbb{R}}\prod_{j=1}^p L_j(1,x)dx = \alpha\big([0,1]^p\big)\end{aligned}$$

where the equality follows from (2.2.16).

We summarize our discussion into the following theorem.

Theorem 5.2.3 *When $d=1$,*

$$n^{-\frac{p+1}{2}}\sum_{x\in\mathbb{Z}} l^p(n,x) \xrightarrow{d} \sigma^{-(p-1)}\int_{\mathbb{R}} L^p(1,x)dx \tag{5.2.8}$$

$$n^{-\frac{p+1}{2}} I_n \xrightarrow{d} \sigma^{-(p-1)}\alpha\big([0,1]^p\big) \tag{5.2.9}$$

In the next section, the weak law in (5.2.9), together with its multi-dimensional version, will be proved by an alternative approach.

We now come to the weak laws for

$$R_n = \#\big\{S[1,n]\big\} \quad \textit{and} \quad J_n = \#\big\{S_1[1,n]\cap\cdots\cap S_p[1,n]\big\}.$$

Notice that in the special case of simple random walk in which

$$\mathbb{P}\{S(k)-S(k-1)=\pm 1\}=\frac{1}{2} \qquad k=1,2,\cdots$$

we have that

$$R_n = \max_{1\le k\le n} S(k) - \min_{1\le k\le n} S(k).$$

By the classic invariance principle we have that

$$R_n/\sqrt{n} \xrightarrow{d} \max_{0\le t\le 1} W(t) - \min_{0\le t\le 1} W(t)$$

Things get a little complicated in the general case, because of the "holes" of unvisited sites in the integer interval $[\min_{1\le k\le n} S(k), \max_{1\le k\le n} S(k)]$. The following theorem indicates that these hole is asymptotically insignificant. We include it without proof, for which the reader is referred to [117].

Theorem 5.2.4 *When $d=1$,*

$$R_n/\sqrt{n} \xrightarrow{d} \sigma\Big(\max_{0\le t\le 1} W(t) - \min_{0\le t\le 1} W(t)\Big) \tag{5.2.10}$$

$$J_n/\sqrt{n} \xrightarrow{d} \sigma\Big(\min_{1\le j\le p}\max_{0\le t\le 1} W_j(t) - \max_{0\le j\le p}\min_{0\le t\le 1} W_j(t)\Big). \tag{5.2.11}$$

Compared to the weak laws (Theorem 5.3.4, Theorem 5.4.3 below), the behavior of J_n and R_n in $d=1$ follows a completely different mechanism, which is not the main focus of this book. In addition to [117], the interested reader is referred to [27] for the large and small deviations established in this case.

5.3 Mutual intersection in sub-critical dimensions

The objectives of this section are the weak laws for the mutual intersection local time

$$I_n = \#\Big\{(k_1,\cdots,k_p)\in[1,n]^p;\quad S_1(k_1)=\cdots=S_p(k_p)\Big\}$$

and for the intersection of independent ranges

$$J_n = \#\big\{S_1[1,n]\cap\cdots\cap S_p[1,n]\big\}$$

generated by the independent, identically distributed and symmetric random walks $S_1(n),\cdots,S_p(n)$ on $\mathbb{Z}^d$ with $d\geq 2$. It has been known for more than 50 years that (Dvoretsky-Erdös [63] and Dvoretsky-Erdös-Kakutani [64], [65]) that the trajectories of $S_1(n),\cdots,S_p(n)$ intersect infinitely often, i.e.,

$$\#\Big\{S_1[1,\infty)\cap\cdots\cap S_p[1,\infty)\Big\}=\infty \qquad a.s.$$

if and only if $p(d-2)\leq d$. Among these cases, the sub-critical dimensions defined by $p(d-2)<d$ and critical dimensions defined by $p(d-2)=d$ are drastically different. Our discussion in this section is for the sub-critical dimensions $p(d-2)<d$.

The difference also appears between cases $d=1$ and $d\geq 2$. Indeed, the limiting distribution in the form given in Theorem 5.2.1 is not well defined in the case $d\geq 2$.

Fix $\epsilon>0$ and define

$$I_{n,\epsilon}=C_{n,\epsilon}^{-1}\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\sum_{k=1}^n p_\epsilon\Big(\frac{\Gamma^{-1/2}\big(S_j(k)-x\big)}{\sqrt{n}}\Big) \tag{5.3.1}$$

where the $d\times d$ positive matrix $\Gamma^{-1/2}$ is a solution of the matrix equation $A^2=\Gamma^{-1}$ (the role of $\Gamma^{-1/2}$ is to canonicalize the random walk), and where

$$C_{n,\epsilon}=\bigg(\sum_{x\in\mathbb{Z}^d}p_\epsilon\Big(\frac{\Gamma^{-1/2}x}{\sqrt{n}}\Big)\bigg)^p\sim n^{\frac{dp}{2}}\det(\Gamma)^{p/2}\qquad (n\to\infty).$$

Applying the invariance principle to the continuous function

$$\widetilde{\Psi}(f_1,\cdots,f_p)=\int_{\mathbb{R}^d}\bigg[\prod_{j=1}^p\int_0^1 p_\epsilon\Big(\Gamma^{-1/2}f_j(s)-x\Big)\bigg]dx$$

on the functional space $\bigotimes_{j=1}^p D\big\{[0,1],\mathbb{R}^d\big\}$ we obtain that

$$\begin{aligned}
&n^{-p}\int_{\mathbb{R}^d}\bigg[\prod_{j=1}^p\sum_{k=1}^n p_\epsilon\Big(\frac{\Gamma^{-1/2}S_j(k)}{\sqrt{n}}-x\Big)\bigg]dx\\
&\xrightarrow{d}\int_{\mathbb{R}^d}\bigg[\prod_{j=1}^p\int_0^1 p_\epsilon\big(W_j(s)-x\big)ds\bigg]dx\\
&=\alpha_\epsilon\big([0,1]^p\big).
\end{aligned}$$

as $n \to \infty$. Here we recall that process $\alpha_\epsilon\big([0,t]^p\big)$ is introduced in Section 2.2.

By continuity of $p_\epsilon(\cdot)$, on the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\bigg[\prod_{j=1}^p\sum_{k=1}^n p_\epsilon\Big(\frac{\Gamma^{-1/2}S_j(k)}{\sqrt{n}}-x\Big)\bigg]dx\\
&= n^{-d/2}\det(\Gamma)^{-1}\int_{\mathbb{R}^d}\bigg[\prod_{j=1}^p\sum_{k=1}^n p_\epsilon\Big(\frac{\Gamma^{-1/2}(S_j(k)-x)}{\sqrt{n}}\Big)\bigg]dx\\
&= n^{-d/2}\det(\Gamma)^{-1}\bigg\{o(1)+\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\sum_{k=1}^n p_\epsilon\Big(\frac{\Gamma^{-1/2}(S_j(k)-x)}{\sqrt{n}}\Big)\bigg\}.
\end{aligned}$$

Therefore, for each $\epsilon > 0$

$$n^{-\frac{2p-d(p-1)}{2}} I_{n,\epsilon} \xrightarrow{d} \det(\Gamma)^{-\frac{p-1}{2}}\alpha_\epsilon\big([0,1]^p\big) \tag{5.3.2}$$

as $n \to \infty$.

The key step toward establishing the weak law for I_n is the following lemma.

Lemma 5.3.1 *Assume that $p(d-2) < d$ and that the random walks are aperiodic.*

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty} n^{-\big(2p-d(p-1)\big)}\mathbb{E}\big[I_n - I_{n,\epsilon}\big]^2 = 0.$$

Proof. The main idea is Fourier transformation. Write

$$I_n(x_1,\cdots,x_{p-1}) = \sum_{k_1,\cdots,k_p=1}^n\prod_{j=1}^{p-1}1_{\{S_j(k_j)-S_p(k_p)=x_j\}}\qquad x_1,\cdots,x_{p-1}\in\mathbb{Z}^d.$$

We have that

$$\begin{aligned}
&\widehat{I}_n(\lambda_1,\cdots,\lambda_{p-1}) \qquad\qquad (5.3.3)\\
&\equiv \sum_{x_1,\cdots,x_{p-1}\in\mathbb{Z}^d} I_n(x_1,\cdots,x_{p-1})\exp\Big\{i\sum_{j=1}^{p-1}\lambda_j\cdot x_j\Big\}\\
&= \sum_{k_1,\cdots,k_p=1}^n\prod_{j=1}^{p-1}\exp\big\{i\lambda_j\cdot\big(S_j(k_j)-S_p(p)\big)\big\}\\
&= \prod_{j=1}^{p-1}\sum_{k=1}^n e^{i\lambda_j\cdot S_j(k)}.
\end{aligned}$$

Here, and elsewhere in the proof, we adopt the convention that

$$\lambda_p = -(\lambda_1 + \cdots + \lambda_{p-1}).$$

Taking Fourier inversion ((B.4), Appendix),

$$I_n = I_n(0) = \frac{1}{(2\pi)^{d(p-1)}} \int_{([-\pi,\pi]^d)^{p-1}} \prod_{j=1}^{p-1} \sum_{k=1}^{n} e^{i\lambda_j \cdot S_j(k)} d\lambda_1 \cdots d\lambda_{p-1}. \quad (5.3.4)$$

Write

$$g_{n,\epsilon}(x_1, \cdots, x_{p-1}) = C_{n,\epsilon}^{-1} \sum_{x\in\mathbb{Z}^d} p_\epsilon\Big(\frac{\Gamma^{-1/2}x}{\sqrt{n}}\Big) \prod_{j=1}^{p-1} p_\epsilon\Big(\frac{\Gamma^{-1/2}(x_j - x)}{\sqrt{n}}\Big) \quad (5.3.5)$$

where $x_1, \cdots, x_{p-1} \in \mathbb{Z}^d$ and $C_{n,\epsilon} > 0$ is given in (5.1.2). One can see that $g_{n,\epsilon}$ is a probability density on $\mathbb{Z}^{d(p-1)}$ and that

$$\begin{aligned} I_{n,\epsilon} &= \sum_{x_1,\cdots,x_p\in\mathbb{Z}^d} g_{n,\epsilon}(x_1,\cdots,x_{p-1}) I_n(x_1,\cdots,x_{p-1}) \\ &= \frac{1}{(2\pi)^{d(p-1)}} \int_{([-\pi,\pi]^d)^{p-1}} \widehat{g}_{n,\epsilon}(\lambda_1,\cdots,\lambda_{p-1}) \prod_{j=1}^{p-1} \sum_{k=1}^{n} e^{i\lambda\cdot S_j(k)} d\lambda_1\cdots d\lambda_{p-1} \end{aligned}$$

where the second step follows from (5.3.4) and Parseval identity ((B.3), Appendix).

Together with (5.3.4), this leads to

$$\begin{aligned} &\mathbb{E}\big[I_n - I_{n,\epsilon}\big]^2 \\ &= \frac{1}{(2\pi)^{2d(p-1)}} \int_{([-\pi,\pi]^d)^{2(p-1)}} d\lambda_1 d\gamma_1 \cdots d\lambda_{p-1} d\gamma_{p-1} \\ &\times F(\lambda_1,\gamma_1,\cdots,\lambda_{p-1},\gamma_{p-1}) \prod_{j=1}^{p} H_n(\lambda_j,\gamma_j) \end{aligned}$$

where

$$\begin{aligned} &F_{n,\epsilon}(\lambda_1,\gamma_1,\cdots,\lambda_{p-1},\gamma_{p-1}) \\ &= \Big[1 - \widehat{g}_{n,\epsilon}(\lambda_1,\cdots,\lambda_{p-1})\Big]\Big[1 - \widehat{g}_{n,\epsilon}(\gamma_1,\cdots,\gamma_{p-1})\Big], \end{aligned}$$

$$H_n(\lambda,\gamma) = \sum_{j,k=1}^{n} \mathbb{E}\exp\Big\{i(\lambda\cdot S(j) + \gamma\cdot S(k)\Big\}.$$

From the fact that $\widehat{g}_{n,\epsilon}$ is real with $|\widehat{g}_{n,\epsilon}| \le 1$,

$$0 \le F(\lambda_1,\gamma_1,\cdots,\lambda_{p-1},\gamma_{p-1}) \le 4.$$

Write

$$\prod_{j=1}^{p}\big|H_n(\lambda_j,\gamma_j)\big|=\prod_{j=1}^{p}\prod_{\substack{1\le k\le p\\ k\ne j}}\big|H_n(\lambda_k,\gamma_k)\big|^{\frac{1}{p-1}}.$$

By Hölder inequality,

$$\begin{aligned}&\mathbb{E}\big[I_n-I_{n,\epsilon}\big]^2\\ &\le\frac{1}{(2\pi)^{2d(p-1)}}\prod_{j=1}^{p}\bigg\{\int_{([-\pi,\pi]^d)^{2(p-1)}}d\lambda_1d\gamma_1\cdots d\lambda_{p-1}d\gamma_{p-1}\\ &\qquad\times F(\lambda_1,\gamma_1,\cdots,\lambda_{p-1},\gamma_{p-1})\prod_{\substack{1\le k\le p\\ k\ne j}}\big|H_n(\lambda_k,\gamma_k)\big|^q\bigg\}^{1/p}\end{aligned}$$

where $q=p(p-1)^{-1}$ is the conjugate of p.

The factor with the index $j=p$ is nice enough so we leave it along at this step. As for $1\le j\le p-1$,

$$\begin{aligned}&\int_{([-\pi,\pi]^d)^{2(p-1)}}d\lambda_1d\gamma_1\cdots d\lambda_{p-1}d\gamma_{p-1}\\ &\qquad\times F(\lambda_1,\gamma_1,\cdots,\lambda_{p-1},\gamma_{p-1})\prod_{\substack{1\le k\le p\\ k\ne j}}\big|H_n(\lambda_k,\gamma_k)\big|^q\\ &\le 4\int_{([-\pi,\pi]^d)^{2(p-1)}}d\lambda_1d\gamma_1\cdots d\lambda_{p-1}d\gamma_{p-1}\prod_{\substack{1\le k\le p\\ k\ne j}}\big|H_n(\lambda_k,\gamma_k)\big|^q\\ &=4\bigg(\int_{([-\pi,\pi]^d)^2}\big|H(\lambda,\gamma)\big|^q d\lambda d\gamma\bigg)^{p-1}\end{aligned}$$

where the last step follows from the translation invariance

$$\begin{aligned}&\int_{([-\pi,\pi]^d)^2}d\lambda_jd\gamma_j\big|H_n(\lambda_p,\gamma_p)\big|^q\\ &=\int_{([-\pi,\pi]^d)^2}d\lambda_jd\gamma_j\big|H_n(\lambda_1+\cdots+\lambda_{p-1},\gamma_1+\cdots+\gamma_{p-1})\big|^q\\ &=\int_{([-\pi,\pi]^d)^2}d\lambda_jd\gamma_j\big|H(\lambda_j,\gamma_j)\big|^q\end{aligned}$$

and Fubini theorem.

Summarizing our argument,

$$\begin{aligned}
&\mathbb{E}\big[I_n - I_{n,\epsilon}\big]^2 \qquad (5.3.6)\\
&\le \frac{4^{p-1}}{(2\pi)^{2d(p-1)}}\bigg(\int_{([-\pi,\pi]^d)^2}\big|H(\lambda,\gamma)|^q d\lambda d\gamma\bigg)^{\frac{(p-1)^2}{p}}\\
&\times\bigg\{\int_{([-\pi,\pi]^d)^{2(p-1)}} d\lambda_1 d\gamma_1\cdots d\lambda_{p-1}d\gamma_{p-1}\\
&\qquad \times F_{n,\epsilon}(\lambda_1,\gamma_1,\cdots,\lambda_{p-1},\gamma_{p-1})\prod_{j=1}^{p-1}\big|H_n(\lambda_j,\gamma_j)\big|^q\bigg\}^{1/p}.
\end{aligned}$$

We now bound $H_n(\lambda,\gamma)$. Let $\varphi(\lambda)$ be the characteristic function of the i.i.d. increments of the random walks. For $j\le k$,

$$\begin{aligned}
&\mathbb{E}\exp\Big\{i\lambda\cdot S(j)+i\gamma\cdot S(k)\Big\}\\
&=\mathbb{E}\exp\Big\{i(\lambda-\gamma)\cdot S(j)+i\gamma\cdot\big(S(k)-S(j)\big)\Big\}\\
&=\varphi(\lambda-\gamma)^j\varphi(\gamma)^{k-j}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big|H_n(\lambda,\gamma)\big| &\le \bigg(\sum_{k=0}^n|\varphi(\lambda-\gamma)|^k\bigg)\bigg(\sum_{l=0}^n|\varphi(\gamma)|^l\bigg) \qquad (5.3.7)\\
&+\bigg(\sum_{l=0}^n|\varphi(\gamma-\lambda)|^l\bigg)\bigg(\sum_{k=0}^n|\varphi(\lambda)|^l\bigg)\\
&=\xi_1(\lambda,\gamma)+\xi_2(\lambda,\gamma)\qquad\text{(say)}.
\end{aligned}$$

Consequently, by triangular inequality and translation invariance,

$$\int_{([-\pi,\pi]^d)^2}\big|H(\lambda,\gamma)|^q d\lambda d\gamma\le 2^q\bigg\{\int_{[-\pi,\pi]^d}\bigg(\sum_{l=0}^n|\varphi(\lambda)|^l\bigg)^q d\lambda\bigg\}^2. \qquad (5.3.8)$$

To continue, we claim that there is a constant $C>0$ such that for any $n\ge 1$,

$$\sum_{k=0}^n\Big|\varphi\Big(\frac{\lambda}{\sqrt{n}}\Big)\Big|^k\le Cn(1+|\lambda|^2)^{-1}\qquad \lambda\in[-\sqrt{n}\pi,\sqrt{n}\pi]^d. \qquad (5.3.9)$$

Indeed, For $|\lambda|\le 1$, we use the obvious bound

$$\sum_{k=0}^n\Big|\varphi\Big(\frac{\lambda}{\sqrt{n}}\Big)\Big|^k\le n\le 2n(1+|\lambda|^2)^{-1}.$$

For $|\lambda|\ge 1$, by (5.1.1) in Lemma 5.1.1, [1]

$$\sum_{k=0}^{n}\Big|\varphi\Big(\frac{\lambda}{\sqrt{n}}\Big)\Big|^k\le\Big\{1-\Big|\varphi\Big(\frac{\lambda}{\sqrt{n}}\Big)\Big|\Big\}^{-1}\le\Big(1-\exp\Big\{-\frac{\delta|\lambda|^2}{n}\Big\}\Big)^{-1}$$
$$\le Cn|\lambda|^{-2}\le 2Cn(1+|\lambda|^2)^{-1}.$$

Apply the variable substitution $\lambda\mapsto\lambda/\sqrt{n}$ to the integral on the right hand side of (5.3.8). By (5.3.9) and by the fact that $(1+|\lambda|)^{-1}\in\mathcal{L}^q(\mathbb{R}^d)$,

$$\int_{([-\pi,\pi]^d)^2}\big|H(\lambda,\gamma)|^q d\lambda d\gamma=O\Big(n^{2q-d}\Big)\qquad(n\to\infty).\tag{5.3.10}$$

In view of (5.3.6), it remains to show that

$$\lim_{\epsilon\to\infty}\limsup_{n\to\infty}n^{-(2p-d(p-1))}\int_{([-\pi,\pi]^d)^{2(p-1)}}d\lambda_1\cdots d\lambda_{p-1}$$
$$\times F_{n,\epsilon}(\lambda_1,\gamma_1,\cdots,\lambda_{p-1},\gamma_{p-1})\prod_{k=1}^{p-1}|H_n(\lambda_k,\gamma_k)|^q=0.$$

From (5.3.7),

$$\prod_{k=1}^{p-1}|H_n(\lambda_k,\gamma_k)|^q\le 2^q\sum_{j_1,\cdots,j_k=1}^{2}\prod_{k=1}^{p-1}\xi_{j_k}^q(\lambda_k,\gamma_k).$$

Hence, it suffices to show that any $(j_1,\cdots,j_{p-1})\in\{1,2\}^{p-1}$,

$$\lim_{\epsilon\to\infty}\limsup_{n\to\infty}n^{-(2p-d(p-1))}\int_{([-\pi,\pi]^d)^{2(p-1)}}d\lambda_1\cdots d\lambda_{p-1}\tag{5.3.11}$$
$$\times F_{n,\epsilon}(\lambda_1,\gamma_1,\cdots,\lambda_{p-1},\gamma_{p-1})\prod_{j=1}^{p-1}\xi_{j_k}^q(\lambda_k,\gamma_k)=0.$$

Indeed, by the periodicity of the integrand, and by the consequential translation invariance,

$$\int_{[-\pi,\pi]^{2d(p-1)}}d\lambda_1 d\gamma_1\cdots d\lambda_{p-1}d\gamma_{p-1}F_{n,\epsilon}(\lambda_1,\gamma_1,\cdots,\lambda_{p-1},\gamma_{p-1})\prod_{k=1}^{p-1}\xi_{j_k}(\lambda_k,\gamma_k)$$
$$=\int_{[-\pi,\pi]^{2d(p-1)}}d\lambda_1 d\gamma_1\cdots d\lambda_{p-1}d\gamma_{p-1}F_{n,\epsilon}(\lambda_1^*,\gamma_1^*,\cdots,\lambda_{p-1}^*,\gamma_{p-1}^*)$$
$$\times\prod_{k=1}^{p-1}\Big(\sum_{l=0}^{n}|\varphi(\lambda_k)|^l\Big)\Big(\sum_{l=0}^{n}|\varphi(\gamma_k)|^l\Big)$$

[1]This is the only place in the entire proof where we use aperiodicity

where for each $1 \le k \le p-1$, λ_k^* and γ_k^* are linear combinations of λ_k and γ_k. By the maps $\lambda_k \mapsto \lambda_k/\sqrt{n}$ and $\gamma_k \mapsto \gamma_k/\sqrt{n}$ $(1 \le k \le p-1)$, and by (5.3.9), the right hand side is bounded by a constant multiple of

$$
\begin{aligned}
&n^{2p-d(p-1)} \int_{\mathbb{R}^{2d(p-1)}} d\lambda_1 d\gamma_1 \cdots d\lambda_{p-1} d\gamma_{p-1}\\
&\times F_{n,\epsilon}\Big(\frac{\lambda_1^*}{\sqrt{n}}, \frac{\gamma_1^*}{\sqrt{n}}, \cdots, \frac{\lambda_{p-1}^*}{\sqrt{n}}, \frac{\gamma_{p-1}^*}{\sqrt{n}}\Big) \prod_{k=1}^{p-1} (1+|\lambda_k|)^{-q}(1+|\gamma_k|)^{-q}.
\end{aligned}
$$

By the fact that

$$
\begin{aligned}
&\widehat{h}_{n,\epsilon}\Big(\frac{\lambda_1}{\sqrt{n}}, \cdots, \frac{\lambda_{p-1}}{\sqrt{n}}\Big)\\
&\longrightarrow \int_{\mathbb{R}^d} dx p_\epsilon(x) \prod_{j=1}^{p-1} \int_{\mathbb{R}^d} e^{-i\lambda_j\cdot y} p_t(x,y)_\epsilon(y-x)dy\\
&= \exp\Big\{ -\frac{\epsilon}{2}\Big(\sum_{j=1}^{p-1} |\lambda_j|^2 + \Big|\sum_{j=1}^{p-1}\lambda_j\Big|^2\Big\} \qquad (n\to\infty)
\end{aligned}
$$

and by dominated convergence theorem,

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^{2d(p-1)}} d\lambda_1 d\gamma_1 \cdots d\lambda_{p-1} d\gamma_{p-1}\\
&\times F_{n,\epsilon}\Big(\frac{\lambda_1^*}{\sqrt{n}}, \frac{\gamma_1^*}{\sqrt{n}}, \cdots, \frac{\lambda_{p-1}^*}{\sqrt{n}}, \frac{\gamma_{p-1}^*}{\sqrt{n}}\Big) \prod_{k=1}^{p-1} (1+|\lambda_k|)^{-q}(1+|\gamma_k|)^{-q}\\
&= \int_{\mathbb{R}^{2d(p-1)}} d\lambda_1 d\gamma_1 \cdots d\lambda_{p-1} d\gamma_{p-1}\bigg[1 - \exp\Big\{ -\frac{\epsilon}{2}\Big(\sum_{j=1}^{p-1} |\lambda_j^*|^2 + \Big|\sum_{j=1}^{p-1}\lambda_j^*\Big|^2\Big)\Big\}\bigg]\\
&\times\bigg[1 - \exp\Big\{ -\frac{\epsilon}{2}\Big(\sum_{j=1}^{p-1} |\gamma_j^*|^2 + \Big|\sum_{j=1}^{p-1}\gamma_j^*\Big|^2\Big)\Big\}\bigg] \prod_{k=1}^{p-1} (1+|\lambda_k|)^{-q}(1+|\gamma_k|)^{-q}.
\end{aligned}
$$

Letting $\epsilon \to 0^+$ on the both side completes the proof of (5.3.11). $\square$

We are ready to state the weak law for I_n.

Theorem 5.3.2 *Under $p(d-2) < d$,*

$$
n^{-\frac{2p-d(p-1)}{2}} I_n \xrightarrow{d} \det(\Gamma)^{-\frac{p-1}{2}} \alpha\big([0,1]^p\big), \tag{5.3.12}
$$

$$
\lim_{n\to\infty} n^{-(2p-d(p-1))}\mathbb{E}I_n^2 = \det(\Gamma)^{-(p-1)}\mathbb{E}\Big[\alpha\big([0,1]^p\big)^2\Big]. \tag{5.3.13}
$$

Proof. By Theorem 2.2.3, $\alpha_\epsilon\big([0,1]^p\big)$ converges to $\alpha\big([0,1]^p\big)$ in all positive moments. Therefore, (5.3.12) and (5.3.13) are direct consequences of (5.3.2) and Lemma 5.3.1 under the extra assumption of aperiodicity.

We now drop the aperiodicity assumption by the method of resolvent approximation. Due to similarity we only prove (5.3.12). We first recall our construction given in Section 5.1: Let $0 < \theta < 1$ be fixed and recall that the random sequence $\{\delta_k\}$ is a Bernoulli sequence independent of $\{S(k)\}$ such that

$$\mathbb{P}\{\delta_k = 1\} = 1 - \mathbb{P}\{\delta_k = 0\} = \theta.$$

Write $\tau_0 = 0$ and

$$\tau_n = \min\{k > \tau_{n-1};\quad \delta_k = 1\} \qquad n = 1, 2, \cdots .$$

Then $\{\tau_n - \tau_{n-1}\}_{n\geq 1}$ is an i.i.d. sequence with common (geometric) distribution

$$\mathbb{P}\{\tau_1 = k\} = \theta(1-\theta)^{k-1} \qquad k = 1, 2, \cdots .$$

Let $\Big\{\{\delta_k^1\}, \{\tau_k^1\}\Big\}, \cdots, \Big\{\{\delta_k^p\}, \{\tau_k^p\}\Big\}$ be p independent copies of the random system $\Big\{\{\delta_k\}, \{\tau_k\}\Big\}$. Observe that

$$\begin{aligned} \bar{I}_n &\equiv \sum_{k_1,\cdots,k_p=1}^{n} 1_{\{S_1(\tau^1_{k_1}) = \cdots = S_p(\tau^p_{k_p})\}} \qquad (5.3.14)\\ &= \sum_{k_1=1}^{\tau_n^1} \cdots \sum_{k_p=1}^{\tau_n^p} \delta^1_{k_1}\cdots\delta^p_{k_p} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}}. \end{aligned}$$

Applying what we have proved to the aperiodic random walks

$$\{S_1(\tau_k^1)\}, \cdots, \{S_p(\tau_k^p)\}$$

gives

$$n^{-\frac{2p-d(p-1)}{2}} \bar{I}_n \xrightarrow{d} \theta^{\frac{d(p-1)}{2}} \det(\Gamma)^{-\frac{p-1}{2}} \alpha\big([0,1]^p\big).$$

Write

$$t_n^j = \delta_1^j + \cdots + \delta_n^j \qquad n = 1, \cdots \quad j = 1, \cdots, p$$

and

$$t_n' = \min\{t_n^1, \cdots, t_n^p\}, \qquad t_n'' = \max\{t_n^1, \cdots, t_n^p\}.$$

Notice that $\bar{I}_n$ is non-decreasing, and that by the law of large numbers

$$t_n'/n \xrightarrow{p} \theta, \qquad t_n''/n \xrightarrow{p} \theta.$$

So we have

$$n^{-\frac{2p-d(p-1)}{2}}\bar{I}_{t'_n} \xrightarrow{d} \theta^p \det(\Gamma)^{-\frac{p-1}{2}}\alpha\big([0,1]^p\big).$$

$$n^{-\frac{2p-d(p-1)}{2}}\bar{I}_{t''_n} \xrightarrow{d} \theta^p \det(\Gamma)^{-\frac{p-1}{2}}\alpha\big([0,1]^p\big).$$

By the fact that

$$t_n^j = \max\{k\ge 0;\quad \tau_k^j \le n\}\qquad j=1,\cdots,p$$

we have

$$\begin{aligned}&\sum_{k_1=1}^{\tau^1_{t^1_n}}\cdots\sum_{k_p=1}^{\tau^p_{t^p_n}} \delta^1_{k_1}\cdots\delta^p_{k_p} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}}\\ &=\sum_{k_1,\cdots,k_p=1}^{n} \delta^1_{k_1}\cdots\delta^p_{k_p} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}}.\end{aligned}$$

Consequently,

$$\bar{I}_{t'_n} \le \sum_{k_1,\cdots,k_p=1}^{n} \delta^1_{k_1}\cdots\delta^p_{k_p} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}} \le \bar{I}_{t''_n}.$$

Thus,

$$\begin{aligned}&n^{-\frac{2p-d(p-1)}{2}}\sum_{k_1,\cdots,k_p=1}^{n} \delta^1_{k_1}\cdots\delta^p_{k_p} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}}\\ &\xrightarrow{d} \theta^p \det(\Gamma)^{-\frac{p-1}{2}}\alpha\big([0,1]^p\big).\end{aligned}$$

The rest of the proof follows from the fact

$$\sum_{k_1,\cdots,k_p=1}^{n} \delta^1_{k_1}\cdots\delta^p_{k_p} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}} \le I_n$$

and the fact that

$$\mathbb{E}\big(I_n\big)=\theta^{-p}\mathbb{E}\bigg(\sum_{k_1,\cdots,k_p=1}^{n} \delta^1_{k_1}\cdots\delta^p_{k_p} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}}\bigg).$$

where the θ can be arbitrarily close to 1. □

We now establish the weak laws for the range intersection J_n. Our approach is to use I_n to approximate J_n.

Lemma 5.3.3 .

When $d = 2$ and $p \geq 2$,

$$\lim_{n\to\infty} \frac{1}{n^2}\mathbb{E}\Big\{I_n - \Big(\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\Big)^p J_n\Big\}^2 = 0. \tag{5.3.15}$$

When $d = 3$ and $p = 2$,

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big\{I_n - \gamma_S^{-2} J_n\Big\}^2 = 0 \tag{5.3.16}$$

where γ_S is the probability of no returning defined in Remark 5.1.12.

Proof. Due to similarity, we only prove (5.3.15). In view of (5.3.13), it is sufficient to show that

$$\limsup_{n\to\infty}\Big(\frac{(\log n)^p}{n}\Big)^2 \mathbb{E}J_n^2 \leq (2\pi)^{2p}\det(\Gamma)\mathbb{E}\Big[\alpha\big([0,1]^p\big)^2\Big], \tag{5.3.17}$$

$$\liminf_{n\to\infty}\frac{(\log n)^p}{n^2}\mathbb{E}\big(I_nJ_n\big) \geq (2\pi)^p\det(\Gamma)^{-\frac{p-2}{2}}\mathbb{E}\Big[\alpha\big([0,1]^p\big)^2\Big]. \tag{5.3.18}$$

Notice that

$$I_n = \sum_{x\in\mathbb{Z}^2}\prod_{j=1}^p l_j(n,x), \qquad J_n = \sum_{x\in\mathbb{Z}^2}\prod_{j=1}^p 1_{\{T_x^{(j)}\leq n\}\}}. \tag{5.3.19}$$

where $l_j(n,x)$ and $T_x^{(j)}$ are local time and hitting time generated by $\{S_j(k)\}_{k\geq 1}$ in a way same as (5.2.1) and (5.1.7), respectively. Thus

$$\begin{aligned}\mathbb{E}J_n &= \sum_{x,y\in\mathbb{Z}^2}\Big[\mathbb{P}\{T_x\leq n,\ T_y\leq n\}\Big]^p\\ &= \sum_{x\in\mathbb{Z}^2}\Big[\mathbb{P}\{T_x\leq n\}\Big]^p + \sum_{x\neq y}\Big[\mathbb{P}\{T_x\leq n,\ T_y\leq n\}\Big]^p.\end{aligned}$$

Observe that

$$\sum_{x\in\mathbb{Z}^2}\Big[\mathbb{P}\{T_x\leq n\}\Big]^p = \mathbb{E}J_n \leq \mathbb{E}I_n.$$

By (5.3.13), it is negligible. Thus, to prove (5.3.17) it is sufficient to show that

$$\begin{aligned}&\limsup_{n\to\infty}\Big(\frac{(\log n)^p}{n}\Big)^2\sum_{x\neq y}\Big[\mathbb{P}\{T_x\leq n,\ T_y\leq n\}\Big]^p \\ &\leq (2\pi)^{2p}\det(\Gamma)\mathbb{E}\Big[\alpha\big([0,1]^p\big)^2\Big].\end{aligned} \tag{5.3.20}$$

For $x \neq y$, write

$$\begin{aligned}
&\mathbb{P}\{T_x \le n,\ T_y \le n\} \\
&= \sum_{1\le j<k\le n} \mathbb{P}\{T_x = j,\ T_y = k\} + \sum_{1\le k<j\le n} \mathbb{P}\{T_x = j,\ T_y = k\}.
\end{aligned}$$

Notice that for $j < k$,

$$\begin{aligned}
&\mathbb{P}\{T_x = j,\ T_y = k\} \\
&\le \mathbb{P}\Big\{T_x = j, S(j+1) \neq y, \cdots, S(k-1) \neq y, S(k) = y\Big\} \\
&= \mathbb{P}\{T_x = j\}\mathbb{P}\{T_{y-x} = k - j\}.
\end{aligned}$$

Therefore,

$$\sum_{1\le j<k\le n} \mathbb{P}\{T_x = j,\ T_y = k\} \le \sum_{j=1}^{n-1} \mathbb{P}\{T_x = j\}\mathbb{P}\{T_{y-x} \le n - j\}.$$

On the other hand, recall ((5.1.5) and (5.1.6)),

$$g_n \equiv \sum_{k=0}^{n} \mathbb{P}\{S(k) = 0\} \sim \frac{1}{2\pi\sqrt{\det(\Gamma)}} \log n \qquad (n \to \infty). \tag{5.3.21}$$

The key observation is the slow varying property of g_n: $g_{[\epsilon n]} \sim g_n$ for any $\epsilon > 0$.

Given $\epsilon > 0$ and $1 \le j < [(1+2\epsilon)n]$, by (5.1.9) in Lemma 5.1.3,

$$\begin{aligned}
&\sum_{k=1}^{[(1+2\epsilon)n]-j} \mathbb{P}\{S(k) = y - x\} = \sum_{k=1}^{[(1+2\epsilon)n]-j} \mathbb{P}\{T_{y-x} = k\} g_{[(1+2\epsilon)n]-k} \\
&\ge g_{[\epsilon n]}\mathbb{P}\{T_{y-x} \le [(1+\epsilon)n] - j\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{1\le j<k\le[(1+2\epsilon)n]} \mathbb{P}\{S(j) = x\}\mathbb{P}\{S(k-j) = y - x\} \\
&\ge g_{[n\epsilon]} \sum_{j=1}^{[(1+2\epsilon)n]-1} \mathbb{P}\{S(j) = x\}\mathbb{P}\{T_{y-x} \le [(1+\epsilon)n] - j\} \\
&\ge g_{[n\epsilon]} \sum_{j=1}^{[(1+\epsilon)n]-1} \mathbb{P}\{T_{x-y} = j\} \sum_{k=1}^{[(1+\epsilon)n]-j} \mathbb{P}\{S(k) = x\} \\
&\ge g_{[n\epsilon]}^2 \sum_{k=1}^{n-j} \mathbb{P}\{T_{x-y} = j\}\mathbb{P}\{T_x \le n - j\}
\end{aligned}$$

where the last step again follows from (5.1.9) in Lemma 5.1.3.

Summarizing our argument,

$$\begin{aligned}
&\sum_{1\le j<k\le n} \mathbb{P}\{T_x = j,\ T_y = k\} \qquad (5.3.22)\\
&\le g_{[n\epsilon]}^{-2} \sum_{1\le j<k\le [(1+2\epsilon)n]} \mathbb{P}\{S(j) = x\}\mathbb{P}\{S(k-j) = y - x\}\\
&= g_{[n\epsilon]}^{-2} \sum_{1\le j<k\le [(1+2\epsilon)n]} \mathbb{P}\{S(j) = x,\ S(k) = y\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{x\neq y} \Big[\mathbb{P}\{T_x \le n,\ T_y \le n\}\Big]^p\\
&\le g_{[\epsilon n]}^{-2} \sum_{x,\in\mathbb{Z}^2} \Bigg[\sum_{j,k=1}^{[(1+2\epsilon)n]} \mathbb{P}\{S(j) = x,\ S(k) = y\}\Bigg]^p\\
&= g_{[\epsilon n]}^{-2} \mathbb{E} I_{[(1+2\epsilon)n]}^2.
\end{aligned}$$

Hence, (5.3.20) follows from (5.3.13) in Theorem 5.3.2, (5.3.21) and the fact that ϵ can be arbitrarily small.

We now prove (5.3.18). By (5.3.19)

$$\begin{aligned}
\mathbb{E}(I_n J_n) &= \sum_{x,y\in\mathbb{Z}^2} \Big[\mathbb{E}\Big\{1_{\{T_x\le n\}} l(n,y)\Big\}\Big]^p \qquad (5.3.23)\\
&\ge \sum_{x\neq y} \Bigg[\sum_{j,k=1}^{n} \mathbb{P}\{T_x = j,\ S(k) = y\}\Bigg]^p.
\end{aligned}$$

By independence of the random walk increments,

$$\begin{aligned}
&\sum_{1\le j<k\le n} \mathbb{P}\{T_x = j,\ S(k) = y\}\\
&= \sum_{1\le j<k\le n} \mathbb{P}\{T_x = j\}\mathbb{P}\{S(k-j) = y - x\}\\
&= \sum_{k=1}^{n-1} \mathbb{P}\{S(k) = y - x\}\mathbb{P}\{T_x \le n - k\}\\
&\ge g_n^{-1} \sum_{k=1}^{n-1} \mathbb{P}\{S(k) = y - x\} \sum_{j=1}^{n-k} \mathbb{P}\{S(j) = x\}\\
&= g_n^{-1} \sum_{1\le j<k\le n} \mathbb{P}\{S(j) = x,\ S(k) = y\}
\end{aligned}$$

where the inequality follows from (5.1.9) in Lemma 5.1.3.

In addition,

$$\begin{aligned}
&\sum_{1\le k<j\le n}\mathbb{P}\{T_x=j,\ S(k)=y\}\\
&=\sum_{1\le k<j\le n}\mathbb{P}\{T_x\ge k,\ S(k)=y,\ S(k+1)\ne x,\\
&\qquad S(j-1)\ne x,\ S(j)=x\}\\
&=\sum_{1\le k<j\le n}\mathbb{P}\{T_x\ge k,\ S(k)=y\}\mathbb{P}\{T_{x-y}=j-k\}\\
&=\sum_{1\le k<j\le n}\mathbb{P}\{S(k)=y\}\mathbb{P}\{T_{x-y}=j-k\}\\
&-\sum_{1\le k<j\le n}\mathbb{P}\{T_x<k,\ S(k)=y\}\mathbb{P}\{T_{x-y}=j-k\}.
\end{aligned}$$

Using (5.1.9) in Lemma 5.1.3 again, the first term on the right hand side is bounded from below by

$$g_n^{-1}\sum_{1\le k<j\le n}\mathbb{P}\{S(j)=x,\ S(k)=y\}.$$

Summarizing our estimate,

$$\begin{aligned}
&\bigg\{\sum_{x\ne y}\bigg[\sum_{j,k=1}^n\mathbb{P}\{T_x=j,\ S(k)=y\}\bigg]^p\bigg\}^{1/p}\\
&\ge g_n^{-1}\bigg\{\sum_{x\ne y}\bigg[\sum_{j,k=1}^n\mathbb{P}\{S(j)=x,\ S(k)=y\}\bigg]^p\bigg\}^{1/p}\\
&-\bigg\{\sum_{x\ne y}\bigg[\sum_{1\le k<j\le n}\mathbb{P}\{T_x<k,\ S(k)=y\}\mathbb{P}\{T_{x-y}=j-k\}\bigg]^p\bigg\}^{1/p}.
\end{aligned}$$

Notice that

$$\sum_{x\ne y}\bigg[\sum_{j,k=1}^n\mathbb{P}\{S(j)=x,\ S(k)=y\}\bigg]^p=\mathbb{E}I_n^2-\mathbb{E}I_n.$$

By (5.3.13), (5.3.21) and (5.3.23) we will have (5.3.18) if

$$\begin{aligned}
\lim_{n\to\infty}\frac{(\log n)^p}{n^2}\sum_{x\ne y}\bigg[&\sum_{1\le k<j\le n}\mathbb{P}\{T_x<k,\ S(k)=y\}\\
&\times\mathbb{P}\{T_{x-y}=j-k\}\bigg]^p=0.
\end{aligned}\tag{5.3.24}$$

By independence of random walk increments,

$$\begin{aligned}
&\sum_{1\le k<j\le n} \mathbb{P}\{T_x < k,\ S(k) = y\}\mathbb{P}\{T_{x-y} = j-k\} \\
&= \sum_{1\le i<k<j\le n} \mathbb{P}\{T_x = i,\ S(k) = y\}\mathbb{P}\{T_{x-y} = j-k\} \\
&= \sum_{1\le i<k<j\le n} \mathbb{P}\{T_x = i\}\mathbb{P}\{S(k-i) = y-x\}\mathbb{P}\{T_{x-y} = j-k\} \\
&\le \mathbb{P}\{T_x \le n\}\mathbb{P}\{T_{x-y} \le n\}\sum_{k=1}^n \mathbb{P}\{S(k) = y-x\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{x\ne y}\Big[\sum_{1\le k<j\le n} \mathbb{P}\{T_x < k,\ S(k) = y\}\mathbb{P}\{T_{x-y} = j-k\}\Big]^p \\
&\le \sum_{x,y\in\mathbb{Z}^2}\Big[\mathbb{P}\{T_x \le n\}\sum_{1\le k<j\le n} \mathbb{P}\{S(k) = y-x\}\mathbb{P}\{T_{x-y} = j-k\}\Big]^p \\
&= \Big\{\sum_{x\in\mathbb{Z}^2}\Big[\mathbb{P}\{T_x \le n\}\Big]^p\Big\}\sum_{y\in\mathbb{Z}^2}\Big[\mathbb{P}\{T_y \le n\}\sum_{k=1}^n \mathbb{P}\{S(k) = y\}\mathbb{P}\{T_y = j-k\}\Big]^p.
\end{aligned}$$

Observe that

$$\sum_{x\in\mathbb{Z}^d}\Big[\mathbb{P}\{T_x \le n\}\Big]^p = \mathbb{E}J_n = O\Big(\frac{n}{(\log n)^p}\Big) \tag{5.3.25}$$

where the second step follows from (5.3.17). By Cauchy-Schwartz inequality

$$\begin{aligned}
&\sum_{y\in\mathbb{Z}^2}\Big[\mathbb{P}\{T_y \le n\}\sum_{k=1}^n \mathbb{P}\{S(k) = y\}\mathbb{P}\{T_y = j-k\}\Big]^p \\
&\le \Big\{\sum_{y\in\mathbb{Z}^2}\Big[\mathbb{P}\{T_y \le n\}\Big]^{2p}\Big\}^{1/2}\Big\{\sum_{y\in\mathbb{Z}^2}\Big[\sum_{k=1}^n \mathbb{P}\{S(k) = y\}\Big]^{2p}\Big\}^{1/2}.
\end{aligned}$$

Replacing p by $2p$ in (5.3.25) and in (5.3.13) gives, respectively, that

$$\sum_{y\in\mathbb{Z}^2}\Big[\mathbb{P}\{T_y \le n\}\Big]^{2p} = O\Big(\frac{n}{(\log n)^{2p}}\Big)$$

and

$$\sum_{y\in\mathbb{Z}^2}\Big[\sum_{k=1}^n \mathbb{P}\{S(k) = y\}\Big]^{2p} = O(n).$$

We have proved the requested conclusion. □

In view of Theorem 5.3.2, an immediate consequence of Lemma 5.3.3 is the following weak law for J_n.

Theorem 5.3.4 .

When $d=2$ *and* $p\ge 2$,

$$\frac{(\log n)^p}{n}J_n \xrightarrow{d} (2\pi)^p\sqrt{\det(\Gamma)}\alpha\big([0,1]^p\big). \tag{5.3.26}$$

When $d=3$ *and* $p=2$,

$$J_n/\sqrt{n} \xrightarrow{d} \det(\Gamma)^{-1/2}\gamma_S^2\alpha\big([0,1]^p\big). \tag{5.3.27}$$

Proof. Obvious. □

Write

$$F_n^{(p)} = \begin{cases} \displaystyle\sum_{x\in\mathbb{Z}^2}\prod_{j=1}^p\Big[l_j(n,x) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{x\in S_j[1,n]\}}\Big] & d=2,\ p\ge 2 \\ \displaystyle\sum_{x\in\mathbb{Z}^3}\prod_{j=1}^2\Big[l_j(n,x) - \gamma_S^{-1}1_{\{x\in S_j[1,n]\}}\Big] & d=3,\ p=2. \end{cases} \tag{5.3.28}$$

By a slight modification of the proof of Lemma 5.3.3, one can show that

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n^2}\mathbb{E}\big[F_n^{(p)}\big]^2 = 0 \tag{5.3.29}$$

when $d=2$, $p\ge 2$; and that

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\mathbb{E}\big[F_n^{(2)}\big]^2 = 0 \tag{5.3.30}$$

when $d=3$ and $p=2$.

Conrrespondent to $I_{n,\epsilon}$ defined in (5.3.1), set

$$J_{n,\epsilon} = C_{n,\epsilon}^{-1}\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\sum_{y\in S_j[1,n]}p_\epsilon\Big(\frac{\Gamma^{-1/2}(y-x)}{\sqrt{n}}\Big). \tag{5.3.31}$$

A natural question is whether J_n can be approximated by $J_{n,\epsilon}$ in a manner similar to Lemma 5.3.1. The answer is "Yes" and "No". On the one hand, the treatment for Lemma 5.3.1 fails when we move from I_n, and $I_{n,\epsilon}$ to J_n and $J_{n,\epsilon}$. On the other hand, given what we have known now, J_n is approximatable by $J_{n,\epsilon}$.

Set,

$$D_{n,\epsilon}^{(p)} = \sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\left[1_{\{x\in S_j[1,n]\}} - C_{n,\epsilon}^{1/p}\sum_{z\in S_j[1,n]} p_\epsilon\Big(\frac{\Gamma^{-1/2}(z-x)}{\sqrt{n}}\Big)\right]. \tag{5.3.32}$$

The following lemma which will be used in Chapter 8.

Lemma 5.3.5 *Assume that the random walks are aperiodic.*

(1). When $d=2$ and $p\geq 2$,

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{(\log n)^{2p}}{n^2}\mathbb{E}\big[D_{n,\epsilon}^{(p)}\big]^2 = 0, \tag{5.3.33}$$

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{(\log n)^{2p}}{n^2}\mathbb{E}\big[J_n - J_{n,\epsilon}\big]^2 = 0. \tag{5.3.34}$$

(2). When $d=3$ and $p=2$,

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\mathbb{E}\big[D_{n,\epsilon}^{(2)}\big]^2 = 0, \tag{5.3.35}$$

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\mathbb{E}\big[J_n - J_{n,\epsilon}\big]^2 = 0. \tag{5.3.36}$$

Proof. Due to similarity, we only consider the case $d=2$, $p\geq 2$. Write

$$\begin{aligned} F_{n,\epsilon}^{(p)} = C_{n,\epsilon}^{-1}\sum_{x\in\mathbb{Z}^2}\prod_{j=1}^p\Bigg[\sum_{k=1}^n p_\epsilon\Big(\frac{\Gamma^{-1/2}(S_j(k)-x)}{\sqrt{n}}\Big) \\ - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\sum_{z\in S_j[1,n]} p_\epsilon\Big(\frac{\Gamma^{-1/2}(z-x)}{\sqrt{n}}\Big)\Bigg]. \end{aligned} \tag{5.3.37}$$

By Lemma 5.3.1 and Lemma 5.3.3 and (5.3.29), it suffice to show that

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n^2}\mathbb{E}\big[F_{n,\epsilon}^{(p)}\big]^2 = 0. \tag{5.3.38}$$

By independence,

$$
\begin{aligned}
&\mathbb{E}\big[F_{n,\epsilon}^{(p)}\big]^2\\
&= C_{n,\epsilon}^{-2}\sum_{x,y\in\mathbb{Z}^2}\bigg[\mathbb{E}\bigg\{\sum_{k=1}^n p_\epsilon\Big(\frac{\Gamma^{-1/2}(S(k)-x)}{\sqrt{n}}\Big)-\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\\
&\times\sum_{z\in S[1,n]}p_\epsilon\Big(\frac{\Gamma^{-1/2}(z-x)}{\sqrt{n}}\Big)\bigg\}\bigg\{\sum_{k=1}^n p_\epsilon\Big(\frac{\Gamma^{-1/2}(S(k)-x)}{\sqrt{n}}\Big)\\
&-\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\sum_{z\in S[1,n]}p_\epsilon\Big(\frac{\Gamma^{-1/2}(z-y)}{\sqrt{n}}\Big)\bigg\}\bigg]^p\\
&= C_{n,\epsilon}^{-2}\sum_{x,y\in\mathbb{Z}^2}\bigg[\sum_{z_1,z_2\in\mathbb{Z}^2}p_\epsilon\Big(\frac{\Gamma^{-1/2}(z_1-x)}{\sqrt{n}}\Big)p_\epsilon\Big(\frac{\Gamma^{-1/2}(z_2-y)}{\sqrt{n}}\Big)\\
&\times\mathbb{E}\bigg\{l(n,z_1)-\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_{z_1}\le n\}}\bigg\}\bigg\{l(n,z_2)-\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_{z_2}\le n\}}\bigg\}\bigg]^p.
\end{aligned}
$$

By Jensen inequality,

$$
\begin{aligned}
&\mathbb{E}\bigg[I_{n,\epsilon}-\Big(\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\Big)^pJ_{n,\epsilon}\bigg]^2\\
&\le C_{n,\epsilon}^{-2/p}\sum_{x,y\in\mathbb{Z}^2}\sum_{z_1,z_2\in\mathbb{Z}^2}p_\epsilon\Big(\frac{\Gamma^{-1/2}(z_1-x)}{\sqrt{n}}\Big)p_\epsilon\Big(\frac{\Gamma^{-1/2}(z_2-y)}{\sqrt{n}}\Big)\\
&\times\bigg|\mathbb{E}\bigg\{l(n,z_1)-\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_{z_1}\le n\}}\bigg\}\bigg\{l(n,z_2)-\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_{z_2}\le n\}}\bigg\}\bigg|^p.
\end{aligned}
$$

By translation invariance, the right hand side is equal to

$$
\sum_{z_1,z_2\in\mathbb{Z}^2}\bigg|\mathbb{E}\bigg\{l(n,z_1)-\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_{z_1}\le n\}}\bigg\}\bigg\{l(n,z_2)-\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_{z_2}\le n\}}\bigg\}\bigg|^p.
$$

When p is even, this is equal to $\mathbb{E}\big[F_n^{(p)}\big]^2$. Consequently, (5.3.38) follows from (5.3.29).

When p is odd, by the inequality

$$
\sum_{z_1,z_2\in\mathbb{Z}^2}|\cdots|^p\le\Big\{\sum_{z_1,z_2\in\mathbb{Z}^2}[\cdots]^{p-1}\Big\}^{1/2}\Big\{\sum_{z_1,z_2\in\mathbb{Z}^2}[\cdots]^{p+1}\Big\}^{1/2}
$$

we obtain

$$
\mathbb{E}\big[F_{n,\epsilon}^{(p)}\big]^2\le\Big\{\mathbb{E}\big[F_n^{(p-1)}\big]^2\Big\}^{1/2}\Big\{\mathbb{E}\big[F_n^{(p+1)}\big]^2\Big\}^{1/2}
$$

Again, (5.3.38) follows from (5.3.29). □

In the rest of the section, $S(k)$ is a symmetric readom walk on $\mathbb{Z}^d$ and we view $S_1(k), \cdots, S_p(k)$ as its independent copies. Recall our notation $R_n = \#\big\{S(1), \cdots, S(n)\big\}$.

By (5.1.9) in Lemma 5.1.3 and by (5.1.6), (5.1.10), when $d \geq 2$ (in which case g_n is slowly increasing),

$$\mathbb{E}R_n = \sum_{x\in\mathbb{Z}^2} \mathbb{P}\{T_x \leq n\} \sim g_n^{-1} \sum_{x\in\mathbb{Z}^2}\sum_{k=1}^n \mathbb{P}\{S(k) = x\} \tag{5.3.39}$$

$$= g_n^{-1} n = \begin{cases} 2\pi\sqrt{\det(\Gamma)}\dfrac{n}{\log n} & d = 2 \\[2ex] \gamma_S n & d \geq 3. \end{cases}$$

Theorem 5.3.6 *Let $f(x)$ be a bounded continuous function on $\mathbb{R}^d$.*

(1). When $d = 2$

$$\frac{1}{n}\bigg\{\sum_{k=1}^n f\Big(\frac{S(k)}{\sqrt{n}}\Big) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\sum_{x\in S[1,n]} f\Big(\frac{x}{\sqrt{n}}\Big)\bigg\} \xrightarrow{P} 0. \tag{5.3.40}$$

Consequently,

$$\bigg(\frac{\log n}{2\pi n\sqrt{\det(\Gamma)}}\sum_{x\in S[1,n]} f\Big(\frac{x}{\sqrt{n}}\Big),\ \ \frac{S(n)}{\sqrt{n}}\bigg) \tag{5.3.41}$$

$$\xrightarrow{d} \bigg(\int_0^1 f\big(\Gamma^{1/2}W(t)\big)dt,\ W(1)\bigg) \quad (n\to\infty).$$

(2). When $d = 3$,

$$\frac{1}{n}\bigg\{\sum_{k=1}^n f\Big(\frac{S(k)}{\sqrt{n}}\Big) - \gamma_S^{-1}\sum_{x\in S[1,n]} f\Big(\frac{x}{\sqrt{n}}\Big)\bigg\} \xrightarrow{P} 0. \tag{5.3.42}$$

Consequently,

$$\bigg(\frac{1}{\gamma_S n}\sum_{x\in S[1,n]} f\Big(\frac{x}{\sqrt{n}}\Big),\ \ \frac{S(n)}{\sqrt{n}}\bigg) \xrightarrow{d} \bigg(\int_0^1 f\big(\Gamma^{1/2}W(t)\big)dt,\ W(1)\bigg) \tag{5.3.43}$$

as $n \to \infty$.

Proof. Due to similarity we only deal with $d = 2$. By invariance principle, (5.3.41) is a direct consequence of (5.3.40).

For each $M > 0$, define $f_M(x) = f(x)$ if $|x| \le M$, and 0 if $|x| > M$. By the classic fact that the random sequence $\max\limits_{1\le k\le n} |S(k)|/\sqrt{n}$ is bounded in probability, all we need is to show that for any fixed $M > 0$

$$\lim_{n\to\infty} \frac{1}{n^2}\mathbb{E}\Big[\sum_{k=1}^n f_M\Big(\frac{S(k)}{\sqrt{n}}\Big) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\sum_{x\in S[1,n]} f_M\Big(\frac{x}{\sqrt{n}}\Big)\Big]^2 = 0. \quad (5.3.44)$$

Take $d = p = 2$ and let $F_n^{(2)}$ be defined in (5.3.28).

$$\begin{aligned}
&\mathbb{E}\big[F_n^{(2)}\big]^2\\
&= \sum_{x,y\in\mathbb{Z}^2}\Big[\mathbb{E}\Big\{l(n,x) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_x\le n\}}\Big\}\Big\{l(n,y) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_y\le n\}}\Big\}\Big]^2.
\end{aligned}$$

By Cauchy-Schwartz inequality,

$$\begin{aligned}
&\Big\{\sum_{x\in\mathbb{Z}^2}\Big|f_M\Big(\frac{x}{\sqrt{n}}\Big)\Big|^2\Big\}\Big\{\mathbb{E}\big[F_n^{(2)}\big]^2\Big\}^{1/2}\\
&\ge \sum_{x,y\in\mathbb{Z}^2} f_M\Big(\frac{x}{\sqrt{n}}\Big)f_M\Big(\frac{y}{\sqrt{n}}\Big)\mathbb{E}\Big\{l(n,x) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_x\le n\}}\Big\}\\
&\qquad\qquad \times\Big\{l(n,y) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}1_{\{T_y\le n\}}\Big\}\\
&= \mathbb{E}\Big[\sum_{x\in\mathbb{Z}^2} f_M\Big(\frac{x}{\sqrt{n}}\Big)l(n,x) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\sum_{x\in\mathbb{Z}^2} f_M\Big(\frac{x}{\sqrt{n}}\Big)1_{\{T_y\le n\}}\Big]^2\\
&= \mathbb{E}\Big[\sum_{k=1}^n f_M\Big(\frac{S(k)}{\sqrt{n}}\Big) - \frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\sum_{x\in S[1,n]} f_M\Big(\frac{x}{\sqrt{n}}\Big)\Big]^2.
\end{aligned}$$

Therefore, the requested (5.3.44) follows from (5.3.29) (with $p = 2$) and the fact that

$$\sum_{x\in\mathbb{Z}^2}\Big|f_M\Big(\frac{x}{\sqrt{n}}\Big)\Big|^2 \sim n\int_{\mathbb{R}^2}|f_M(x)|^2dx$$

as $n\to\infty$. $\square$

Theorem 5.3.6 will play an important role in Chapter 7.

5.4 Self-intersection in dimension two

The goal of this section is to establish the weak laws for the double self-intersection local time

$$Q_n = \sum_{1\le j<k\le n} 1_{\{S(j)=S(k)\}}$$

and for the range

$$R_n = \#\big\{S[1,n]\big\} = \#\big\{S(1),\cdots,S(n)\big\}$$

of the planar symmetric random walk $S(k)$.

By (5.1.6), it is straightforward to see that

$$\mathbb{E}Q_n = \sum_{1\le j<k\le n} \mathbb{P}\{S(k-j)=0\} \sim \frac{1}{2\pi\sqrt{\det(\Gamma)}} n\log n. \tag{5.4.1}$$

The asymptotical expansion of R_n is given in (5.3.39). Taking $f\equiv 1$ in Theorem 5.3.6 leads to

$$\lim_{n\to\infty} R_n/\mathbb{E}R_n = 1.$$

in probability.

This type of concentration pehnomenon happens also to Q_n in multi-dimensions. The right question is to ask what can be said about the limit laws for $Q_n-\mathbb{E}Q_n$ and for $R_n-\mathbb{E}R_n$. We start our investigation by estimating the variance of Q_n and R_n.

Lemma 5.4.1

$$\mathrm{Var}\,(Q_n) = O(n^2) \qquad (n\to\infty), \tag{5.4.2}$$

$$\mathrm{Var}\,(R_n) = O\Big(\frac{n^2}{(\log n)^4}\Big) \qquad (n\to\infty). \tag{5.4.3}$$

Proof. Due to similarity, we only prove (5.4.3). For each integer $k\ge 1$, write

$$a_k = \sup\Big\{\mathrm{Var}\,(R_n)^{1/2};\quad 2^k < n \le 2^{k+1}\Big\}.$$

For each $n\in(2^k, 2^{k+1}]$ let $n_1 = [n/2]$ and $n_2 = n - n_1$. Consider the decomposition

$$R_n = R_{n_1} + \widetilde{R}_{n_2} - \#\big\{S[1,n_1]\cap S[n_1+1,n]\big\}$$

where $\widetilde{R}_{n_2} = \#\big\{S[n_1+1, n]\big\}$ is independent of R_{n_1} and of the distribution same as R_{n_2}, and

$$\#\big\{S[1,n_1]\cap S[n_1,n]\big\} \overset{d}{=} \#\big\{S[0,n_1-1]\cap \widetilde{S}[1,n_2]\big\}$$

where $\widetilde{S}(n)$ is an independent copy of $S(n)$. By Cauchy-Schwartz inequality,

$$\begin{aligned}&\mathrm{Var}\,(R_n)^{1/2}\\ &\le \Big(\mathrm{Var}\,(R_{n_1}) + \mathrm{Var}\,(R_{n_2})\Big)^{1/2} + \mathrm{Var}\,\Big(\#\big\{S[0,n_1-1]\cap \widetilde{S}[1,n_2]\big\}\Big)^{1/2}.\end{aligned}$$

Taking $p=2$ in (5.3.17) gives that the second term on the right hand side is bounded by $C_1k^{-2}2^k$ for some constant independent of k. Thus,

$$a_k \le 2^{1/2}a_{k-1} + C_1k^{-2}2^k \qquad k=2,3,\cdots.$$

Write $b_k = k^2 2^{-k}a_k$. Then

$$b_k \le \theta b_{k-1} + C_1$$

for any $\theta > 2^{-1/2}$ and sufficiently large k. Consequently, the sequence b_k is bounded. $\square$

Recall that γ_t is the renormalized self-intersection local time

$$\begin{aligned}\gamma\big([0,t]^2_<\big) = &\int\!\!\int_{\{0\le r<s\le t\}} \delta_0\big(W(r)-W(s)\big)drds\\ &-\mathbb{E}\int\!\!\int_{\{0\le r<s\le t\}} \delta_0\big(W(r)-W(s)\big)drds\end{aligned}$$

run by a planar Brownian motion $W(t)$.

Theorem 5.4.2

$$n^{-1}\Big\{Q_n - \mathbb{E}Q_n\Big\} \xrightarrow{d} \det(\Gamma)^{-1/2}\gamma\big([0,1]^2_<\big).$$

Proof. By the method of resolvent approximation, we may assume that the random walk is aperiodic.

The proof is based on a same type of triangular approximation used in Theorem 2.4.1.

Let $N\ge 1$ be a large integer and write

$$\begin{aligned}&\mathcal{A}_l^k(n) = \Big(\frac{2l}{2^{k+1}}n, \frac{2l+1}{2^{k+1}}n\Big] \times \Big(\frac{2l+1}{2^{k+1}}n, \frac{2l+2}{2^{k+1}}n\Big]\\ &l=0,1,\cdots,2^k-1 \quad k=0,1,\cdots N-1.\end{aligned}$$

$$\mathcal{B}_l(n) = \Big\{(i,j);\quad \frac{l-1}{2^N}n < i < j \le \frac{l}{2^N}n\Big\} \qquad l = 1, \cdots, 2^N.$$

It is helpful for the reader to check Figure 2.1 in Section 2.4 for visualization of these sets.

We have

$$Q_n = \sum_{k=1}^{N-1}\sum_{l=1}^{2^k-1}\sum_{(i,j)\in\mathcal{A}_l^k(n)} 1_{\{S(i)=S(j)\}} + \sum_{l=1}^{2^N}\sum_{(i,j)\in\mathcal{B}_l(n)} 1_{\{S(i)=S(j)\}}. \tag{5.4.4}$$

Observe that the sequence

$$\sum_{(i,j)\in\mathcal{B}_l(n)} 1_{\{S(i)=S(j)\}} \qquad l = 1, \cdots, 2^N$$

is independent. Therefore,

$$\begin{aligned}
&\mathrm{Var}\Big\{\sum_{l=1}^{2^N}\sum_{(i,j)\in\mathcal{B}_l(n)} 1_{\{S(i)=S(j)\}}\Big\} \\
&= \sum_{l=1}^{2^N}\mathrm{Var}\Big\{\sum_{(i,j)\in\mathcal{B}_l(n)} 1_{\{S(i)=S(j)\}}\Big\} \\
&= O\Big(2^{-N}n^2\Big).
\end{aligned}$$

In particular,

$$\lim_{N\to\infty}\limsup_{n\to\infty} n^{-2}\mathrm{Var}\Big\{\sum_{l=1}^{2^N}\sum_{(i,j)\in\mathcal{B}_l(n)} 1_{\{S(i)=S(j)\}}\Big\} = 0. \tag{5.4.5}$$

Suppose that we can prove that as $n \to \infty$,

$$\frac{1}{n}\sum_{k=1}^{N-1}\sum_{l=1}^{2^k-1}\sum_{(i,j)\in\mathcal{A}_l^k(n)} 1_{\{S(i)=S(j)\}} \xrightarrow{d} \det(\Gamma)^{-1/2}\beta(D_N) \tag{5.4.6}$$

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\Big\{\sum_{k=1}^{N-1}\sum_{l=1}^{2^k-1}\sum_{(i,j)\in\mathcal{A}_l^k(n)} 1_{\{S(i)=S(j)\}}\Big\} = \det(\Gamma)^{-1/2}\mathbb{E}\beta(D_N). \tag{5.4.7}$$

for every N, where $\beta(\cdot)$ is the self-intersection local time (without being renormalization)

$$\beta(A) = \iint_A \delta_0\big(W(r) - W(s)\big)drds \qquad A \subset (\mathbb{R}^+)^2_<$$

discussed in Section 2.3,

$$D_N = \bigcup_{k=1}^{N-1} \bigcup_{l=1}^{2^k-1} A_l^k$$

and the sets A_l^k are defined in (2.4.2). By Theorem 2.4.1, $\beta(D_N) - \mathbb{E}\beta(D_N)$ converges to γ_1 as $N \to \infty$. Therefore the desired conclusion holds.

It remains to prove (5.4.6) and (5.4.7). Due to similarity in proof, we only deal with (5.4.6). Fix $\epsilon > 0$ and apply the invariance principle to the continuous functional

$$\Psi(f) = \iint_{D_N} p_{2\epsilon}\Big(\Gamma^{-1//2}\big(f(r) - f(s)\big)\Big)drds$$

on $D\big\{[0,1], \mathbb{R}^2\big\}$ gives that

$$\frac{1}{n^2}\sum_{k=1}^{N-1}\sum_{l=1}^{2^k-1}\sum_{(i,j)\in\mathcal{A}_l^k(n)} p_{2\epsilon}\Big(\frac{\Gamma^{-1/2}\big(S(i)-S(j)\big)}{\sqrt{n}}\Big)$$
$$\xrightarrow{d} \iint_{D_N} p_{2\epsilon}\big(W(r) - W(s)\big)drds = \beta_\epsilon(D_N)$$

as $n \to \infty$, where the random measure

$$\beta_\epsilon(A) = \int_{\mathbb{R}^d}\Big[\iint_A p_\epsilon\big(W(r)-x\big)p_\epsilon\big(W(s)-x\big)drds\Big]dx \qquad A \subset (\mathbb{R}+)^2_<$$

was constructed and discussed (in a more general form) in Section 2.3 and, it was shown that $\beta_\epsilon(A)$ converges to the double self intersection $\beta(A)$ at $A = D_N$ in L_2-moment as $\epsilon \to 0^+$.

From above discussion, the proof of (5.4.6) has been reduced to show that for any fixed $1 \le k \le N-1$ and $l = 0, 1, \cdots 2^k - 1$,

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n^2}\mathbb{E}\bigg\{\sum_{(i,j)\in\mathcal{A}_l^k(n)} 1_{\{S(i)=S(j)\}} \tag{5.4.8}$$
$$- \frac{1}{\sqrt{\det(\Gamma)}n}\sum_{(i,j)\in\mathcal{A}_l^k(n)} p_{2\epsilon}\Big(\frac{\Gamma^{-1/2}\big(S(i)-S(j)\big)}{\sqrt{n}}\Big)\bigg\}^2 = 0.$$

Indeed, there are $t_n, \tilde{t}_n = \frac{n}{2^k} \pm 1$ such that

$$\sum_{(i,j)\in\mathcal{A}_l^k(n)} 1_{\{S(i)=S(j)\}} - \frac{1}{\sqrt{\det(\Gamma)}n}\sum_{(i,j)\in\mathcal{A}_l^k(n)} p_{2\epsilon}\Big(\frac{\Gamma^{-1/2}\big(S(i)-S(j)\big)}{\sqrt{n}}\Big)$$
$$\stackrel{d}{=} \sum_{i=0}^{t_n}\sum_{j=1}^{\tilde{t}_n} 1_{\{S(i)=\widetilde{S}(j)\}} - \frac{1}{\sqrt{\det(\Gamma)}n}\sum_{i=0}^{t_n}\sum_{j=1}^{\tilde{t}_n} p_{2\epsilon}\Big(\frac{\Gamma^{-1/2}\big(S(i)-\widetilde{S}(j)\big)}{\sqrt{n}}\Big)$$

where $\widetilde{S}(n)$ is an independent copy of $S(n)$. By the continuity of $p_t(x)$ with respect to both t and x, the right hand side has the expansion, in the notation used in (5.3.1) (with $d = p = 2$),

$$o(n) + I_{t_n} - I_{t_n, 2^k\epsilon} \qquad (n \to \infty)$$

Hence, (5.4.8) follows from Lemma 5.3.1. □

Theorem 5.4.3 .

$$\mathbb{E}\Big\{(Q_n - \mathbb{E}Q_n) + \Big(\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\Big)^2 (R_n - \mathbb{E}R_n)\Big\}^2 = o(n^2) \tag{5.4.9}$$

as $n \to \infty$. In particular,

$$\frac{(\log n)^2}{n}(R_n - \mathbb{E}R_n) \xrightarrow{d} -(2\pi)^2\sqrt{\det(\Gamma)}\gamma_1. \tag{5.4.10}$$

Proof. Correspondent to (5.4.4)

$$\begin{aligned} R_n &= \sum_{l=1}^{2^N} \#\Big\{S\Big(\frac{l-1}{2^N}n, \frac{l}{2^N}n\Big]\Big\} \\ &\quad - \sum_{k=1}^{N-1}\sum_{l=1}^{2^k-1} \#\Big\{S\Big(\frac{2l}{2^{k+1}}n, \frac{2l+1}{2^{k+1}}n\Big] \cap S\Big(\frac{2l+1}{2^{k+1}}n, \frac{2l+2}{2^{k+1}}n\Big]\Big\}. \end{aligned}$$

By an argument similar to the one used for (5.4.5),

$$\lim_{N\to\infty}\limsup_{n\to\infty}\frac{(\log n)^4}{n^2}\mathrm{Var}\left\{\sum_{l=1}^{2^N}\#\Big\{S\Big(\frac{l-1}{2^N}n, \frac{l}{2^N}n\Big]\Big\}\right\} = 0.$$

To prove (5.4.9), therefore, it is sufficient to prove that for fixed k, l,

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{n^2}\mathbb{E}\bigg\{\sum_{(i,j)\in\mathcal{A}_l^k(n)} 1_{\{S(i)=S(j)\}} \\ &-\Big(\frac{\log n}{2\pi\sqrt{\det(\Gamma)}}\Big)^2 \#\Big\{S\Big(\frac{2l}{2^{k+1}}n, \frac{2l+1}{2^{k+1}}n\Big] \cap S\Big(\frac{2l+1}{2^{k+1}}n, \frac{2l+2}{2^{k+1}}n\Big]\Big\}\bigg\}^2 \\ &= 0. \end{aligned}$$

This clearly follows from (5.3.15) in Lemma 5.3.3 with $d = p = 2$.

The statement (5.4.10) is a direct consequence of Theorem 5.4.2 and (5.4.9). □

5.5 Self-intersection in high dimensions

For the mutual intersection, the words "high dimensional" here really mean the critical dimensions consists of the case $d = 4$, $p = 2$ and the case $d = p = 3$. In these cases, the Brownian intersection local time does not exist. Naturally, the weak law for I_n and J_n, if any, will take a form different from those given in Theorem 5.3.2 and Theorem 5.3.4. Here we include, without giving the proof, the following theorem of Le Gall [118] on the weak convergence for I_n and J_n.

Theorem 5.5.1 *(1) When $d = 4$ and $p = 2$,*

$$(\log n)^{-1} I_n \xrightarrow{d} (2\pi)^{-2} \det(\Gamma)^{-1/2} U^2 \tag{5.5.1}$$

$$(\log n)^{-1} J_n \xrightarrow{d} \gamma_S^2 (2\pi)^{-2} \det(\Gamma)^{-1/2} U^2 \tag{5.5.2}$$

where U is a $N(0,1)$ random variable.

(2) When $d = p = 3$,

$$(\log n)^{-1} I_n \xrightarrow{d} (2\pi)^{-2} \frac{1}{\det(\Gamma)} V \tag{5.5.3}$$

$$(\log n)^{-1} J_n \xrightarrow{d} (2\pi)^{-2} \frac{\gamma_S^3}{\det(\Gamma)} V \tag{5.5.4}$$

where the random variable V has the gamma distribution with parameter $1/4$.

As for the self-intersection, "high dimension" means that $d \geq 3$, partially because the Brownian self-intersection local time does not exist (even by renormalization). To establish the weak laws for $Q_n - \mathbb{E}Q_n$ and $R_n - \mathbb{E}R_n$, we first investigate the asymptotic orders of their variances.

Lemma 5.5.2 *When $d = 3$,*

$$\mathrm{Var}\,(Q_n) \sim \frac{1}{2\pi^2 \det(\Gamma)} n \log n \qquad (n \to \infty), \tag{5.5.5}$$

$$\mathrm{Var}\,(R_n) \sim \frac{\gamma_S^2}{2\pi^2 \det(\Gamma)} n \log n \qquad (n \to \infty). \tag{5.5.6}$$

When $d \geq 4$

$$\mathrm{Var}\,(Q_n) \sim \mathfrak{D}^2 n \qquad (n \to \infty) \tag{5.5.7}$$

$$\mathrm{Var}\,(R_n) \sim \gamma_S^2 \mathfrak{D}^2 n \qquad (n \to \infty) \tag{5.5.8}$$

where the limiting variance $\mathfrak{D}^2$ is given by

$$\mathfrak{D}^2 = 3A^2 + A + 2B$$

and

$$A = \sum_{k=1}^{\infty} \mathbb{P}\{S_k = 0\} \quad \textit{and} \quad B = \sum_{x \in \mathbb{Z}^d} \left[\sum_{k=1}^{\infty} \mathbb{P}\{S_k = x\}\right]^3$$

Proof. First, we have to show that $B < \infty$ so that the statement for $d \geq 4$ is well posted. More generally, we show that in the super-critical dimensions defined by $p(d-2) > d$,

$$\sum_{x \in \mathbb{Z}^3} \sum_{x \in \mathbb{Z}^d} \left[\sum_{k=1}^{\infty} \mathbb{P}\{S_k = x\}\right]^p < \infty \tag{5.5.9}$$

By the argument of resolvent approximation we may do it under aperiodicity. Indeed, under aperiodicity, it is known ((1.4), [151]) that

$$\sum_{k=1}^{\infty} \mathbb{P}\{S_k = x\} \leq \sum_{z \in \mathbb{Z}^d} \frac{\pi(z)}{1 + |z - x|^{d-2}} \qquad \mathbb{Z}^d$$

where $\pi(x)$ is a summable non-negative function on $\mathbb{Z}^d$. By triangular inequality,

$$\begin{aligned}
&\left\{\sum_{x \in \mathbb{Z}^3} \sum_{x \in \mathbb{Z}^d} \left[\sum_{k=1}^{\infty} \mathbb{P}\{S_k = x\}\right]^p\right\}^{1/p} \\
&\leq \sum_{z \in \mathbb{Z}^d} \pi(z) \left\{\sum_{x \in \mathbb{Z}^d} \frac{1}{(1 + |z - x|^{d-2})^p}\right\}^{1/p} \\
&= \left\{\sum_{z \in \mathbb{Z}^d} \pi(z)\right\}\left\{\sum_{x \in \mathbb{Z}^d} \frac{1}{(1 + |x|^{d-2})^p}\right\}^{1/p} < \infty
\end{aligned}$$

Due to similarity, we only compute the variance for Q_n. That is, we prove (5.5.5) and (5.5.6). We introduce the following notations: We write $P_k(x) = \mathbb{P}\{S(k) = x\}$ and set $\{\widetilde{S}(k)\}$ for an independent copy of $\{S(k)\}$. Let

$$\mathfrak{G}(x) = \sum_{k=1}^{\infty} P_k(x) \quad \text{and} \quad \mathfrak{G}_k(x) = \sum_{j=1}^{k} P_j(x)$$

be the Green's function and partial Green's function, respectively.

We begin with the decomposition

$$Q_n = \sum_{j=1}^{n-1}\sum_{k=j+1}^{\infty} 1_{\{S_k=S_j\}} - \sum_{j=1}^{n-1}\sum_{k=n+1}^{\infty} 1_{\{S_k=S_j\}} \tag{5.5.10}$$
$$= \sum_{j=1}^{n-1} Z_j - \sum_{j=1}^{n-1} W_j^n \qquad (say).$$

For any $1 \le i \le j \le k \le n$,

$$\begin{aligned}
\mathbb{E}W_i^n W_j^n &= \sum_{k,l=n+1}^{\infty} \mathbb{P}\{S_l - S_j = 0,\ S_k - S_j = S_i - S_j\}\\
&= \sum_{k,l=n-j+1}^{\infty} \mathbb{P}\{S_l = 0,\ S_k = \widetilde{S}_{j-i}\}\\
&= \sum_{x\in\mathbb{Z}^d} P_{j-i}(x) \sum_{k,l=n-j+1}^{\infty} \mathbb{P}\{S_l = 0,\ S_k = x\}\\
&\le \sum_{x\in\mathbb{Z}^d} P_{j-i}(x) \sum_{n-j+1\le k\le l<\infty} \mathbb{P}\{S_l = 0,\ S_k = x\}\\
&\quad + \sum_{x\in\mathbb{Z}^d} P_{j-i}(x) \sum_{n-j+1\le l\le k<\infty} \mathbb{P}\{S_l = 0,\ S_k = x\}.
\end{aligned}$$

For the first term on the right hand side,

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d} P_{j-i}(x) \sum_{n-j+1\le k\le l<\infty} \mathbb{P}\{S_l = 0,\ S_k = x\}\\
&= \sum_{x\in\mathbb{Z}^d} P_{j-i}(x) \sum_{n-j+1\le k\le l<\infty} \mathbb{P}\{S_{l-k} = x\}\mathbb{P}\{S_k = x\}\\
&\le C \sum_{x\in\mathbb{Z}^d} P_{j-i}(x) \sum_{n-j+1\le k\le l<\infty} P_{l-k}(x)\frac{1}{k^{d/2}}\\
&= C \sum_{n-j+1\le k\le l<\infty} P_{l-k+j-i}(0)\frac{1}{k^{d/2}}
\end{aligned}$$

where the second step follows from classic fact that $\sup_{x\in\mathbb{Z}^d} P_k(x) = O(k^{-d/2})$.

As for the second term, a similar estimate yields that

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d} P_{j-i}(x) \sum_{n-j+1\le l\le k<\infty} \mathbb{P}\{S_l = 0,\ S_k = -x\}\\
&\le C \sum_{n-j+1\le l\le k<\infty} P_{k-l+j-i}(0)\frac{1}{l^{d/2}}.
\end{aligned}$$

Hence,

$$\mathbb{E}W_i^n W_j^n \le 2C \sum_{n-j+1\le k\le l<\infty} p_{l-k+j-i}(0)\frac{1}{l^{d/2}}$$
$$= 2\Big(\sum_{k=n-j+1}^{\infty} p_k(0)\Big)\Big(\sum_{l=j-i}^{\infty}\frac{1}{l^{d/2}}\Big) = O\Big((n-j)^{1-\frac{d}{2}}(j-i)^{1-\frac{d}{2}}\Big).$$

Therefore, as $n\to\infty$,

$$\sum_{j=1}^{n}\sum_{i=1}^{j}\mathbb{E}W_i^n W_j^n = \begin{cases} O(n) & d=3 \\ O\Big((\log n)^2\Big) & d=4 \\ O(1) & d\ge 5. \end{cases} \tag{5.5.11}$$

For $1\le i\le j\le n$,

$$\mathrm{Cov}\,(Z_i, Z_j) = \mathrm{Cov}\left(\sum_{k=1}^{\infty} 1_{\{S_k=\widetilde{S}_{j-i}\}},\ \sum_{k=1}^{\infty} 1_{\{S_k=0\}}\right) \tag{5.5.12}$$
$$= \sum_{k,l=1}^{\infty}\Big\{\mathbb{P}\{S_k=\widetilde{S}_{j-i},\ S_l=0\} - P_{k+j-i}(0)P_l(0)\Big\}$$
$$= \sum_{x\in\mathbb{Z}^d} P_{j-i}(x)\sum_{k,l=1}^{\infty}\mathbb{P}\{S_k=-x,\ S_l=0\} - G(0)\sum_{k=j-i+1}^{\infty} P_k(0).$$

Write

$$\sum_{x\in\mathbb{Z}^d} P_{j-i}(x)\sum_{k,l=1}^{\infty}\mathbb{P}\{S_k=x,\ S_l=0\}$$
$$= \sum_{x\in\mathbb{Z}^d} P_{j-i}(x)\sum_{1\le k\le l<\infty}\mathbb{P}\{S_k=x\ S_l=0\}$$
$$+ \sum_{x\in\mathbb{Z}^d} P_{j-i}(x)\sum_{1\le l<k<\infty}\mathbb{P}\{S_k=x\ S_l=0\}.$$

The first term on the right hand side is equal to

$$\sum_{x\in\mathbb{Z}^d} P_{j-i}(x)\sum_{1\le k\le l<\infty} P_k(x)p_{l-k}(x)$$
$$= \sum_{x\in\mathbb{Z}^d} P_{j-i}(x)\mathfrak{G}(x)\sum_{l=0}^{\infty} P_l(x)$$
$$= p_{j-i}(0)\mathfrak{G}(0)\big(1+\mathfrak{G}(0)\big) + \sum_{x\ne 0} P_{j-i}(x)\mathfrak{G}^2(x).$$

As for the second term, it is equal to

$$\sum_{x\in\mathbb{Z}^d} P_{j-i}(x) \sum_{1\le l<k<\infty} P_{k-l}(x)P_l(0)$$
$$= \sum_{1\le l<k<\infty} P_{k-l+j-i}(0)P_l(0) = \mathfrak{G}(0) \sum_{k=j-i+1}^{\infty} P_k(0).$$

In summary of the argument since (5.5.12),

$$\text{Cov}\,(Z_i, Z_j) = P_{j-i}(0)\mathfrak{G}(0)\big(1+\mathfrak{G}(0)\big) + \sum_{x\ne 0} P_{j-i}(x)\mathfrak{G}^2(x).$$

Consequently,

$$\text{Var}\Big(\sum_{i=1}^n Z_i\Big) = \sum_{i=1}^n Var(Z_i) + 2\sum_{1\le i<j\le n} Cov(Z_i, Z_j) \tag{5.5.13}$$
$$= n\mathfrak{G}(0)\big(1+\mathfrak{G}(0)\big) + 2\mathfrak{G}(0)\big(1+\mathfrak{G}(0)\big)\sum_{j=1}^{n-1}\mathfrak{G}_{n-j}(0)$$
$$+ 2\sum_{x\ne 0}\mathfrak{G}^2(x)\sum_{i=1}^{n-1}\mathfrak{G}_{n-i}(x)$$
$$= n\mathfrak{G}(0)\big(1+\mathfrak{G}(0)\big) + 2\mathfrak{G}(0)\sum_{j=1}^{n-1}\mathfrak{G}_j(0) + 2\sum_{x\in\mathbb{Z}^d}\mathfrak{G}^2(x)\sum_{j=1}^{n-1}\mathfrak{G}_j(x).$$

When $d\ge 4$, by (5.5.9

$$\sum_{x\in\mathbb{Z}^d}\mathfrak{G}^3(x) < \infty.$$

By (5.5.13),

$$\text{Var}\Big(\sum_{i=1}^n Z_i\Big) \sim n\Big\{\mathfrak{G}(0)\big(1+\mathfrak{G}(0)\big) + 2\mathfrak{G}^2(0) + 2\sum_{x\in\mathbb{Z}^d}\mathfrak{G}^3(x)\Big\}$$

as $n\to\infty$. By (5.5.10) and (5.5.11), this implies (5.5.7).

We now consider the case $d=3$. We use the fact that (p. 308, [147])

$$\mathfrak{G}(x) \sim (2\pi)^{-1}\det(\Gamma)^{-1/2}\langle x, \Gamma^{-1}x\rangle^{-1/2} \qquad (|x|\to\infty).$$

Thus,

$$\sum_{\langle x,\Gamma^{-1}x\rangle > j}\mathfrak{G}^2(x)\mathfrak{G}_j(x) = O\Big(\frac{1}{j}\sum_{x\in\mathbb{Z}^d}\mathfrak{G}_j(x)\Big) = O(1) \qquad (j\to\infty). \tag{5.5.14}$$

In addition,

$$\begin{aligned}
&\sum_{\langle x,\Gamma^{-1}x\rangle\le j}\mathfrak{G}^2(x)\big(\mathfrak{G}(x)-\mathfrak{G}_j(x)\big) \qquad (5.5.15)\\
&\le\big(\mathfrak{G}(0)-\mathfrak{G}_j(0)\big)\sum_{\langle x,\Gamma^{-1}x\rangle\le j}\mathfrak{G}^2(x)\\
&\le C\big(\mathfrak{G}(0)-\mathfrak{G}_j(0)\big)\sum_{\langle x,\Gamma^{-1}x\rangle\le j}\frac{1}{1+|x|^2}=O(1)\qquad (j\to\infty)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\langle x,\Gamma^{-1}x\rangle\le j}\mathfrak{G}^3(x)\sim\frac{1}{(2\pi)^3\det(\Gamma)^3}\int_{\{1\le\langle x,\Gamma^{-1}x\rangle\le j\}}\langle x,\Gamma^{-1}x\rangle^{-3/2}dx \qquad (5.5.16)\\
&=\frac{1}{(2\pi)^3\det(\Gamma)}\int_{\{1\le|y|\le\sqrt{j}\}}|y|^{-3}dy=\frac{1}{(2\pi)^2\det(\Gamma)}\log j\qquad (j\to\infty).
\end{aligned}$$

Combining (5.5.14), (5.5.15) and (5.5.16) gives

$$\sum_{x\in\mathbb{Z}^d}\mathfrak{G}^2(x)\mathfrak{G}_j(x)\sim\frac{1}{(2\pi)^2\det(\Gamma)}\log j\qquad (j\to\infty).$$

By (5.5.12),

$$\mathrm{Var}\Big(\sum_{i=1}^n Z_i\Big)\sim\frac{2}{(2\pi)^2\det(\Gamma)}n\log n\qquad (n\to\infty)$$

which, together with (5.5.10) and (5.5.11), implies (5.5.5). □

The following theorem shows that $Q_n-\mathbb{E}Q_n$ and $R_n-\mathbb{E}R_n$ are attracted by normal distributions as $d\ge 3$.

Theorem 5.5.3 *(1). When $d=3$,*

$$\frac{1}{\sqrt{n\log n}}\Big(Q_n-\mathbb{E}Q_n\Big)\xrightarrow{d}N\Big(0,\ \frac{1}{2\pi^2\det(\Gamma)}\Big),\tag{5.5.17}$$

$$\frac{1}{\sqrt{n\log n}}\Big(R_n-\mathbb{E}R_n\Big)\xrightarrow{d}N\Big(0,\ \frac{\gamma_S^2}{2\pi^2\det(\Gamma)}\Big).\tag{5.5.18}$$

(2). When $d\ge 4$,

$$\frac{1}{\sqrt{n}}\Big(Q_n-\mathbb{E}Q_n\Big)\xrightarrow{d}N(0,\mathfrak{D}^2),\tag{5.5.19}$$

$$\frac{1}{\sqrt{n}}\Big(R_n - \mathbb{E}R_n\Big) \xrightarrow{d} N(0, \gamma_S^2\mathfrak{D}^2), \tag{5.5.20}$$

where $\mathfrak{D}^2$ is defined in Lemma 5.5.2.

Proof. Due to similarity we only prove (5.5.17). Let $\{\gamma_n\}$ be a positive sequence such that

$$\gamma_n \longrightarrow \infty \quad \text{and} \quad \gamma_n = o\Big(\sqrt{\log n}\Big) \qquad (n \to \infty).$$

Let $0 = n_0 < n_1 < \cdots < n_{\gamma_n} = n$ be an integer partition of $[0, n]$ such that for each $1 \le i \le \gamma_n$, $n - i - n_{i-1} = [n\gamma_n^{-1}]$ or $[n\gamma_n^{-1}] + 1$. Then

$$Q_n = \sum_{i=1}^{\gamma_n} \sum_{n_{i-1}<j<k\le n_i} 1_{\{S(j)=S(k)\}} + \sum_{i=1}^{\gamma_n-1} \sum_{j=n_{i-1}+1}^{n_i} \sum_{k=n_i+1}^{n} 1_{\{S(j)=S(k)\}}. \tag{5.5.21}$$

For each $1 \le i \le \gamma_n - 1$,

$$\sum_{j=n_{i-1}+1}^{n_i} \sum_{k=n_i+1}^{n} 1_{\{S(j)=S(k)\}} \overset{d}{=} \sum_{j=1}^{n_i-n_{i-1}} \sum_{k=1}^{n-n_i} 1_{\{S_j=\widetilde{S}(k)\}} \le \sum_{j=1}^{n}\sum_{k=1}^{n} 1_{\{S(j)=\widetilde{S}(k)\}}$$

where $\{\widetilde{S}(k)\}$ is an independent copy of $\{S(k)\}$. By (5.3.13) in Theorem 5.3.2 (with $d = 2$ and $p = 2$),

$$\begin{aligned} &\mathbb{E}\bigg(\sum_{i=1}^{\gamma_n-1} \sum_{j=n_{i-1}+1}^{n_i} \sum_{k=n_i+1}^{n} 1_{\{S(j)=S(k)\}}\bigg)^2 \\ &\le \gamma_n^2\mathbb{E}\bigg(\sum_{j=1}^{n}\sum_{k=1}^{n} 1_{\{S(j)=\widetilde{S}(k)\}}\bigg)^2 = o(n\log n). \end{aligned} \tag{5.5.22}$$

In addition, notice that the random variables

$$\sum_{n_{i-1}<j<k\le n_i} 1_{\{S(j)=S(k)\}} \qquad i = 1, 2, \cdots, \gamma_n$$

are independent with

$$\sum_{n_{i-1}<j<k\le n_i} 1_{\{S_j=S_k\}} \overset{d}{=} Q_{n_i-n_{i-1}} \qquad i = 1, 2, \cdots, \gamma_n.$$

Comparing (5.5.5) and (5.5.22)

$$\mathrm{Var}\left(\sum_{i=1}^{\gamma_n}\sum_{n_{i-1}<j<k\le n_i} 1_{\{S_j=S_k\}}\right)\sim \mathfrak{D}^2 n\log n.$$

From Theorem 6.4.1 in the next chapter, we can check Lindbergh condition. Hence,

$$\sum_{i=1}^{\gamma_n}\left\{\sum_{n_{i-1}<j<k\le n_i} 1_{\{S_j=S_k\}} - \mathbb{E}\Big(\sum_{n_{i-1}<j<k\le n_i} 1_{\{S_j=S_k\}}\Big)\right\}\Big/ n\log n$$
$$\xrightarrow{d} N\Big(0,\ \frac{1}{2\pi^2\det(\Gamma)}\Big).$$

Finally, (5.5.17) follows from (5.5.21) and (5.5.22). $\square$

From the argument we put for Theorem 5.5.3, we see that as $d\ge 3$, the short range self-intersection dominates the long-range self-intersection. Consequently, $Q_n-\mathbb{E}Q_n$ (as well as $R_n-\mathbb{E}R_n$) is approximated by independent sums. That is the reason we have Gaussian laws here.

5.6 Notes and comments

Section 5.1

We refer the reader to the book by Spitzer [147] for the collection of general theory on random walks. The terminology used by Spitzer is slightly different from what we use here: What we call aperiodic is strongly aperiodic in Spitzer [147].

Section 5.2

Theorem 5.2.1, Theorem 5.2.3 and the argument used to their establishment come from the paper by Chen and Li ([31]). The proof of Lemma 5.2.2 is essentially adopted from Jain and Pruitt [97].

Section 5.3

The weak laws for I_n and J_n was first studied by Le Gall [117] in the sub-critical dimensions. He shows that every integral moment of properly normalized I_n and J_n converges to the same moment of $\alpha\big([0,1]^p\big)$. Prior to [117], Le Gall had obtained a analogue for the intersection of two independent Brownian sausages ([116]). The idea of using Fourier transformation in proving the weak law for I_n (Theorem 5.3.2) can be found in the paper by

Rosen [143]. The idea of approximating J_n by I_n (proof of Lemma 5.3.3) can be seen in the paper by Le Gall and Rosen ([124]).

Exercise 1. This exercise is related to Theorem 5.3.2. Take $p = 2$ and $d \le 3$.

1. Directly prove that $\mathbb{E}I_n \sim cn^{\frac{4-d}{2}}$ and find out the constant c.

2. Give a shorter proof of Theorem 5.3.2 (That is, to simplify the proof of Lemma 5.3.1 or establish something else instead) in this special case.

Section 5.4.

The weak laws for $Q_n - \mathbb{E}Q_n$ and $R_n - \mathbb{E}R_n$ in the case $d = 2$ was first investigated by Le Gall [117] and his results were essentially what we state in Theorem 5.4.2 and in Theorem 5.4.3. Le Gall's work was later extended by Rosen [143] and by Le Gall and Rosen [124] to more general random walks, for example, to the stable random walks.

The study of the range in $d = 2$ has been a sensetional and a lot of interesting things happen in this dimension. Let, for example, $R_n^{(p)}$ be the number of distinct sites visited exactly p ($p \ge 1$) times by $S(k)$ in the first n steps. Hamana ([83]) points out that

$$\frac{(\log n)^3}{n}\Big[R_n^{(p)} - \mathbb{E}R_n^{(p)}\Big] \xrightarrow{d} -16\pi^3 \det(\Gamma)^2 \gamma_1.$$

If $\widehat{R}_n^{(p)}$ is the number of distinct sites visited at least p ($p \ge 1$) times by $S(k)$ in the first n steps, then

$$\frac{(\log n)^3}{n}\Big[\widehat{R}_n^{(p)} - \mathbb{E}\widehat{R}_n^{(p)}\Big] \xrightarrow{d} -4\pi^2 \det(\Gamma)^2 \gamma_1.$$

A stricking fact is that these limit laws does not depend on p at all! This seems to suggest that once being visited, then it costs almost nothing to be visited again.

When p increases with n, the points in $\widehat{R}_n^{(p)}$ are called *thick points*. A remarkable progress in recent years is the investigation of thick points led by Dembo, Peres, Rosen and Zeitouni ([45]) claiming that as $p_n \sim a(\log n)^2$,

$$\lim_{n\to\infty} (\log n)^{-1} \log \widehat{R}_n^{(p_n)} = 1 - \pi a \qquad a.s.$$

A direct consequence of their work is the final solution of the long standing Erdös-Taylor conjecture.

Other related models are the time needed for the random walk to cover a big disk (Cover-time problem) and the waiting time for visiting a fresh site after first n steps. These models were investigated by Dembo, Peres and Rosen ([46]).

Section 5.5.

Theorem 5.3.6 was established in [25].

Section 5.6.

Due to some historic reasons, much of the focus is on the $R_n - \mathbb{E}R_n$ as $d \geq 3$. The central limit theorem given in (5.5.20) was first obtained by Jain and Orey ([94]) in the special case $d \geq 5$. Later by Jain and Pruitt ([95]), (5.5.20) was established for $d = 4$ and (5.5.18) was established for $d = 3$. Similar results were obtained by Yor ([164]) and by Calais and Yor ([20]) in the setting of Brownian sausages. The central limit theorem for $Q_n - \mathbb{E}Q_n$ (given in Theorem 5.5.3) was much later established in Chen ([30]). The argument used in Lemma 5.5.2 goes back at least to Jain and Orey ([94]).

The following exercise appears as an application of the weak laws for Q_n stated in Theorem 5.2.3 (with $p = 2$), Theorem 5.4.2 and Theorem 5.5.3. The reader is referred to Kesten and Spitzer ([100]), Bolthausen ([13]), Khoshnevisan and Lewis ([104]).

Exercise 2. Let $\{\xi(x);\quad x \in \mathbb{Z}^d\}$ be a family of i.i.d. random variables independent of the random walk $\{S(k)\}_{k\geq 1}$ such that $\mathbb{E}\xi(0) = 0$ and $\mathbb{E}\xi^2(0) = 1$. The random sequence

$$\zeta_n = \sum_{k=1}^{n} \xi\big(S(k)\big) \qquad n = 1, 2, \cdots \tag{5.6.1}$$

is called *random walk in random scenery* in literature, where the random scenery is referred to $\{\xi(x);\quad x \in \mathbb{Z}^d\}$.

(1). When $d = 1$,

$$n^{-3/4}\zeta_n \xrightarrow{d} U\sqrt{\frac{1}{\sigma}\int_{-\infty}^{\infty} L^2(1, x)dx} \tag{5.6.2}$$

where $U \sim N(0, 1)$ independent of $W(t)$.

(2) When $d = 2$,

$$(n\log n)^{-1/2}\zeta_n \xrightarrow{d} \frac{1}{\sqrt{2\pi}\sqrt[4]{\det(\Gamma)}}U. \tag{5.6.3}$$

(3) When $d \geq 3$,

$$(n \log n)^{-1/2} \zeta_n \xrightarrow{d} U \sqrt{\sum_{k=1}^{\infty} \mathbb{P}\{S(k) = 0\}}. \tag{5.6.4}$$

Hint: $\zeta_n = \sum_{x \in \mathbb{Z}^d} l(n, x)\xi(x)$.

Chapter 6

Inequalities and integrabilities

-

The large deviation principle appears as asymptotically sharp inequalities for the underline probabilities. On the other hand, sharp inequalities are usually important tools for studying large deviation principle. When such inequalities take a clean and symmetric form, themselves become a focus of interest. Our first goal of this chapter is to establish such inequalities for additive and sub-additive functionals associated with mutual intersections. The key idea in these inequalities is to separate the (mutual) intersections happening in different time periods.

It is hard to image that a practically interesting model satisfying large deviation principle does not have some basic exponential integrabilities. The second goal of this chapter is to establish the exponential integrabilities for the intersection local times I_n, Q_n, the range R_n and the intersection J_n of independent ranges.

The argument by sub-additivity has been an effective approach of establishing large deviations integrabilities.. The multi-nomial inequalities established in Section 6.1 provide an environment that allows such argument. In Section 6.2, they are applied directly to the problem of maximal integrabilities for I_n and J_n.

In Section 6.3 and Section 6.4, the integrabilities established for I_n and J_n are extended to the renormalized self-intersection local time $Q_n - \mathbb{E}Q_n$ and to the renormalized range $R_n - \mathbb{E}R_n$.

6.1 Multi-nomial inequalities

Let $S_1(k), \cdots, S_p(k)$ be independent symmetric random walks in $\mathbb{Z}^d$. For convenience, we view $S_1(k), \cdots, S_p(k)$ as independent copies of the random walk $\{S(k)\}$.

Let the integer $a \geq 2$ and the increasing integers $1 \leq n_1 < \cdots < n_a$ be fixed. We use the notations $\mathcal{F}_k^x(S)$ and $\mathcal{G}_k^x(S)$ ($x \in \mathbb{Z}^d$, $k = 1, \cdots, a$) for two measurable functionals of a random walk $S = \{S(k)\}$ on $\mathbb{Z}^d$. The following assumptions are assumed throughout this section.

(1). Adaptivity of $\mathcal{F}_k^x(S)$: $\mathcal{F}_k^x(S) \in \sigma\{S(j);\ 1 \leq j \leq n_k\}$ for any $x \in \mathbb{Z}^d$, $k = 1, \cdots, a$.

(2) Spatial homogeneity of $\mathcal{G}_k^x(S)$: $\mathcal{G}_k^x(z + S) = \mathcal{G}_k^{x-z}$[1] for any $x, z \in \mathbb{Z}^d$, $k = 1, \cdots, a$.

The functional $\mathcal{F}_k^x(S)$ ($k = 1, \cdots, a$) is said to be *sub-additive* on the increasing sequence $n_1 < n_2 < \cdot < n_a$ with respect to the *increment functional* $\mathcal{G}_k^x$, if

$$\mathcal{F}_1(S)^x = \mathcal{G}_1^x(S) \quad \text{and} \quad \mathcal{F}_k^x(S) \leq \mathcal{F}_{k-1}^x(S) + \mathcal{G}_k^x(S \circ \theta^{n_{k-1}}) \tag{6.1.1}$$

$x \in \mathbb{Z}^d$, $k = 1, \cdots, a$, where θ is the usual shift operator of the Markov chain $\{S(k)\}$: $S(n) \circ \theta = S(n+1)$, so that

$$S \circ \theta^{n_{k-1}} = \Big\{S(n_{k-1}),\ S(n_{k-1}+1), \cdots\Big\}.$$

Further, $\mathcal{F}_k^x(S)$ is said to be *additive* if the inequality in (6.1.1) is replaced by equality.

Theorem 6.1.1 . *Assume the sub-additivity defined in (6.1.1) and assume that $\mathcal{F}_k^x(S) \geq 0$, $\mathcal{G}_k^x(S) \geq 0$ for $x \in \mathbb{Z}^d$, $k = 1, \cdots, a$. For any integer $m \geq 1$,*

$$\begin{aligned} &\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p \mathcal{F}_a^x(S_j)\Big]^m\bigg\}^{1/p} \qquad (6.1.2)\\ &\leq \sum_{\substack{l_1+\cdots+l_a=m\\ l_1,\cdots,l_a\geq 0}} \frac{m!}{l_1!\cdots l_a!}\prod_{k=1}^a \bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p \mathcal{G}_k^x(S_j)\Big]^{l_k}\bigg\}^{1/p}. \end{aligned}$$

[1] Here we allow the random walk to start at non-zero point. In this way, $z+S$ constitutes a random walk for any $z \in \mathbb{Z}^d$.

Consequently, for any $u > 0$,

$$\sum_{m=0}^{\infty}\frac{u^m}{m!}\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{p}\mathcal{F}_a^x(S_j)\Big]^m\bigg\}^{1/p} \tag{6.1.3}$$
$$\le \prod_{k=1}^{a}\sum_{m=0}^{\infty}\frac{u^m}{m!}\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{p}\mathcal{G}_k^x(S_j)\Big]^m\bigg\}^{1/p}.$$

Proof. Obviously, (6.1.3) follows directly from (6.1.2). By an induction procedure, we need only to show that for $k = 2, \cdots, a$,

$$\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{p}\mathcal{F}_k^x(S_j)\Big]^m\bigg\}^{1/p} \tag{6.1.4}$$
$$\le \sum_{l=0}^{m}\binom{m}{l}\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{p}\mathcal{F}_{k-1}^x(S_j)\Big]^l\bigg\}^{1/p}\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{p}\mathcal{G}_k^x(S_j)\Big]^{m-l}\bigg\}^{1/p}$$

Let k be fixed. and write

$$\Delta_1(x) = \mathcal{F}_{k-1}^x(S) \quad\text{and}\quad \Delta_2(x) = \mathcal{G}_k^x(S\circ\theta^{n_{k-1}}).$$

We have

$$\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{p}\mathcal{F}_k^x(S_j)\Big]^m = \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{E}\prod_{l=1}^{m}\mathcal{F}_k^{x_l}(S)\Big]^p$$
$$= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\bigg[\sum_{i_1,\cdots,i_m=1}^{2}\mathbb{E}\Big(\Delta_{i_1}(x_1)\cdots\Delta_{i_m}(x_m)\Big)\bigg]^p.$$

By triangular inequality,

$$\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{p}\mathcal{F}_k^x(S_j)\Big]^m\bigg\}^{1/p}$$
$$\le \sum_{i_1,\cdots,i_m=1}^{2}\bigg\{\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{E}\Big(\Delta_{i_1}(x_1)\cdots\Delta_{i_m}(x_m)\Big)\Big]^p\bigg\}^{1/p}$$
$$= \sum_{l=0}^{m}\binom{m}{l}\bigg\{\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)$$
$$\times\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\Big]^p\bigg\}^{1/p}.$$

All we need is to prove that

$$\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\left[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\right]^p \tag{6.1.5}$$
$$\le \mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\mathcal{F}_{k-1}^x(S_j)\Big]^l\,\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\mathcal{G}_k^x(S_j)\Big]^{m-l}.$$

For fixed $(x_1,\cdots,x_l)$,

$$\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\left[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\right]^p$$
$$=\mathbb{E}\bigg\{\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)$$
$$\times\left[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\right]^{p-1}\bigg\}$$
$$=\mathbb{E}\bigg\{\Big(\prod_{j=1}^l\mathcal{F}_{k-1}^{x_j}(S)\Big)\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\prod_{j=l+1}^m\mathcal{G}_k^{x_j}(S\circ\theta^{n_{k-1}})$$
$$\times\left[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\right]^{p-1}\bigg\}.$$

By spatial homogeneity of $\mathcal{G}_k^x(S)$,

$$\prod_{j=l+1}^m\mathcal{G}_k^{x_j}(S\circ\theta^{n_{k-1}})=\prod_{j=l+1}^m\mathcal{G}_k^{x_j-S(n_{k-1}}(S^{(k)})$$

where the random walk

$$S^{(k)}(n)=S(n_{k-1}+n)-S(n)\qquad n=1,2,\cdots$$

is independent of $\{S(1),\cdots,S(n_{k-1})\}$.

Write

$$\xi(x_{l+1},\cdots,x_m)=\mathbb{E}\prod_{j=l+1}^m\mathcal{G}_k^{x_j}(S).$$

We have that

$$\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\Big]^p$$
$$=\mathbb{E}\Big\{((\Big(\prod_{j=1}^l\mathcal{F}_{k-1}^{x_j}(S)\Big)\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\xi\big(x_{l+1}-S(n_{k-1}),\cdots,x_m-S(n_{k-1})\big)$$
$$\times\Big[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\Big]^{p-1}\Big\}.$$

By Hölder inequality,

$$\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\xi\big(x_{l+1}-S(n_{k-1}),\cdots,x_m-S(n_{k-1})\big)$$
$$\times\Big[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\Big]^{p-1}$$
$$\le\Big\{\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\xi^p\big(x_{l+1}-S(n_{k-1}),\cdots,x_m-S(n_{k-1})\big)\Big\}^{1/p}$$
$$\times\Big\{\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\Big]^p\Big\}^{\frac{p-1}{p}}.$$

By translation invariance,

$$\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\xi^p\big(x_{l+1}-S(n_{k-1}),\cdots,x_m-S(n_{k-1})\big)$$
$$=\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\xi^p\big(x_{l+1},\cdots,x_m\big)=\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\mathcal{G}_k^x(S_j)\Big]^{m-l}.$$

Summarizing our argument, for any $x_1,\cdots,x_l\in\mathbb{Z}^d$

$$\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\Big]^p$$
$$\le\Big\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\mathcal{G}_k^x(S_j)\Big]^{m-l}\Big\}^{1/p}\Big(\mathbb{E}\prod_{j=1}^l\mathcal{F}_{k-1}^{x_j}(S)\Big)$$
$$\times\Big\{\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\Big]^p\Big\}^{\frac{p-1}{p}}.$$

Hence, we conclude that

$$\sum_{x_{l+1},\cdots,x_m\in\mathbb{Z}^d}\left[\mathbb{E}\Big(\Delta_1(x_1)\cdots\Delta_1(x_l)\Big)\Big(\Delta_2(x_{l+1})\cdots\Delta_2(x_m)\Big)\right]^p$$
$$\le \mathbb{E}\bigg[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\mathcal{G}_k^x(S_j)\bigg]^{m-l}\bigg[\mathbb{E}\prod_{j=1}^l\mathcal{F}_{k-1}^{x_j}(S)\bigg]^p$$

at least for $x_1,\cdots,x_l\in\mathbb{Z}^d$ with

$$\mathbb{E}\prod_{j=1}^l\mathcal{F}_{k-1}^{x_j}(S)>0.$$

In another case, the left hand side is equal to 0. Therefore, above inequality holds for every $(x_1,\cdots,x_l)$.

Finally, the requested (6.1.5) follows from the fact that

$$\sum_{x_1,\cdots,x_l\in\mathbb{Z}^d}\bigg[\mathbb{E}\prod_{j=1}^l\mathcal{F}_{k-1}^{x_j}(S)\bigg]^p=\mathbb{E}\bigg[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\mathcal{F}_{k-1}^x(S_j)\bigg]^l.$$

□

The inequality given in (6.1.2) (and the inequalities in (6.1.13) and (6.1.14) in the next theorem) is called *multi-nomial inequality* due to the multi-nomial form on its right hand side. In this book, the inequalities established in Theorem 6.1.1 are mainly applied to the following three spacial cases.

Case 1. Recall that $l(n,x)$ is the local time of $\{S(k)\}$. The functional $\mathcal{F}_k^x(S)=l(n_k,x)$ $(k=1,\cdots,a)$ is additive on $n_1<\cdots<n_a$ with increment functional $\mathcal{G}_k^x(S)=l(n_k-n_{k-1},x)$ $(k=1,\cdots,a)$. Here we follow the convention $n_0=0$.

Case 2. The functional $\mathcal{F}_k^x(S)=1_{\{x\in S[1,n_k]\}}$ $(k=1,\cdots,a)$ is sub-additive on $n_1<\cdots<n_a$ with the increment functional

$$\mathcal{G}_k^x(S)=1_{\{x\in S[1,n_k-n_{k-1}]\}}\qquad k=1,\cdots,a.$$

Case 3. An additive version of Case 2 is defined by

$$\mathcal{F}_k^x(S)=\sum_{j=1}^k 1_{\{x\in S(n_{j-1},n_j]\}}\qquad k=1,\cdots,a. \tag{6.1.6}$$

it is additive on $n_1<\cdots<n_a$ with the same increment functional $\mathcal{G}_k^x(S)=1_{\{x\in S[1,n_k-n_{k-1}]\}}$.

Applying Theorem 6.1.1 to these three case we obtain the following inequalities.

$$\Big\{\mathbb{E}I_{n_a}^m\Big\}^{1/p} \le \sum_{\substack{l_1+\cdots+l_a=m\\ l_1,\cdots,l_a\ge 0}} \frac{m!}{l_1!\cdots l_a!}\prod_{k=1}^a \Big\{\mathbb{E}I_{n_k-n_{k-1}}^{l_k}\Big\}^{1/p}, \tag{6.1.7}$$

$$\sum_{m=0}^\infty \frac{u^m}{m!}\big(\mathbb{E}I_{n_a}^m\big)^{1/p} \le \prod_{k=1}^a \sum_{m=0}^\infty \frac{\theta^m}{m!}\big(\mathbb{E}I_{n_k-n_{k-1}}^m\big)^{1/p}; \tag{6.1.8}$$

$$\Big\{\mathbb{E}J_{n_a}^m\Big\}^{1/p} \le \sum_{\substack{l_1+\cdots+l_a=m\\ l_1,\cdots,l_a\ge 0}} \frac{m!}{l_1!\cdots l_a!}\prod_{k=1}^a \Big\{\mathbb{E}J_{n_k-n_{k-1}}^{l_k}\Big\}^{1/p}, \tag{6.1.9}$$

$$\sum_{m=0}^\infty \frac{u^m}{m!}\big(\mathbb{E}J_{n_a}^m\big)^{1/p} \le \prod_{k=1}^a \sum_{m=0}^\infty \frac{\theta^m}{m!}\big(\mathbb{E}J_{n_k-n_{k-1}}^m\big)^{1/p}; \tag{6.1.10}$$

$$\Big\{\mathbb{E}\mathcal{J}_{n_a}^m\Big\}^{1/p} \le \sum_{\substack{l_1+\cdots+l_a=m\\ l_1,\cdots,l_a\ge 0}} \frac{m!}{l_1!\cdots l_a!}\prod_{k=1}^a \Big\{\mathbb{E}J_{n_k-n_{k-1}}^{l_k}\Big\}^{1/p}, \tag{6.1.11}$$

$$\sum_{m=0}^\infty \frac{u^m}{m!}\big(\mathbb{E}\mathcal{J}_{n_a}^m\big)^{1/p} \le \prod_{k=1}^a \sum_{m=0}^\infty \frac{\theta^m}{m!}\big(\mathbb{E}J_{n_k-n_{k-1}}^m\big)^{1/p}. \tag{6.1.12}$$

Here we recall that

$$I_n = \sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p l_j(n,x) \quad\text{and}\quad J_n = \sum_{x\in Z^d}\prod_{j=1}^p 1_{\{x\in S_j[1,n]\}}$$

and we define

$$\mathcal{J}_{n_a} = \sum_{x\in Z^d}\prod_{j=1}^p\sum_{k=1}^a 1_{\{x\in S_j(n_{k-1},n_k]\}}.$$

The argument for Theorem 6.1.1 does not work if $\mathcal{F}_k(S)^x$ and $\mathcal{G}_k(S)^x$ are allowed to take negative values. On the other hand, there are some practical cases in which one has to deal with the additive functional with sign flapping. As seen in Section 5.3, an important way to study the limit behavior of the quantity

$$\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p \mathcal{F}_k^x(S_j)$$

is to approximate it by its smoothed version

$$\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p (h*\mathcal{F})_k^x(S_j)$$

where $h(x)$ is a suitably chosen density function on $\mathbb{Z}^d$ and

$$(h*\mathcal{F})_k^x(S)=\sum_{y\in\mathbb{Z}^d} h(x-y)\mathcal{F}_k^y(S).$$

In this scenario, it is desirable to provide inequalities to bound the error term. In this case, the additive (or, sub-additive in general) functional one has to deal with may take form as

$$\mathcal{F}_k^x(S)-(h*\mathcal{F})_k^x(S)$$

which is not constantly non-negative.

In the following discussion, we focus our attention to the case of two independent random walks.

Theorem 6.1.2 *Let $\mathcal{F}_k^x(S)$ $(k=1,\cdots,a)$ be additive on the increasing sequence $1<n_1<\cdots<n_a$ with the increment $\mathcal{G}_k^x$. For any probability density function $h(x)$ on $\mathbb{Z}^d$ and any integer $m\ge 1$,*

$$\Big\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 \mathcal{F}_a^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 (h*\mathcal{F})_a^x(S_j)\Big]^m\Big\}^{1/2} \tag{6.1.13}$$
$$\le \sum_{\substack{l_1+\cdots+l_a=m\\ l_1,\cdots,l_a\ge 0}}\frac{m!}{l_1!\cdots l_a!}\prod_{k=1}^a\Big\{\mathbb{E}\Big|\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 \mathcal{G}_k^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 (h*\mathcal{G})_k^x(S_j)\Big|^{l_k}\Big\}^{1/2},$$

$$\Big\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 \Big(\mathcal{F}_a^x(S_j)-(h*\mathcal{F})_a^x(S_j)\Big)\Big]^m\Big\}^{1/2} \tag{6.1.14}$$
$$\le \sum_{\substack{l_1+\cdots+l_a=m\\ l_1,\cdots,l_a\ge 0}}\frac{m!}{l_1!\cdots l_a!}\prod_{k=1}^a\Big\{\mathbb{E}\Big|\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 \Big(\mathcal{G}_k^x(S_j)-(h*\mathcal{G})_k^x(S_j)\Big)\Big|^{l_k}\Big\}^{1/2}.$$

Consequently, for any $u>0$,

$$\sum_{m=0}^\infty\frac{u^m}{m!}\Big\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 \mathcal{F}_a^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 (h*\mathcal{F})_a^x(S_j)\Big]^m\Big\}^{1/2} \tag{6.1.15}$$
$$\le\prod_{k=1}^a\sum_{m=0}^\infty\frac{u^m}{m!}\Big\{\mathbb{E}\Big|\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 \mathcal{G}_k^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 (h*\mathcal{G})_k^x(S_j)\Big|^m\Big\}^{1/2},$$

$$\sum_{m=0}^{\infty}\frac{u^m}{m!}\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}\Big(\mathcal{F}_a^x(S_j)-(h*\mathcal{F})_a^x(S_j)\Big)\Big]^m\bigg\}^{1/2} \tag{6.1.16}$$
$$\le\prod_{k=1}^{a}\sum_{m=0}^{\infty}\frac{u^m}{m!}\bigg\{\mathbb{E}\Big|\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}\Big(\mathcal{G}_k^x(S_j)-(h*\mathcal{G})_k^x(S_j)\Big)\Big|^m\bigg\}^{1/2}.$$

Proof. Clearly, (6.1.15) and (6.1.16) are direct consequences of (6.1.13) and (6.1.14), respectively.

By additivity (Recall the convention $n_0=0$)

$$\mathcal{F}_a^x(S)=\sum_{k=1}^{a-1}\mathcal{G}_k^x(S\circ\theta^{n_{k-1}}).$$

To prove (6.1.15), by induction procedure, it suffices to show that

$$\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}\mathcal{F}_k^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}(h*\mathcal{F})_k^x(S_j)\Big]^m\bigg\}^{1/2} \tag{6.1.17}$$
$$\le\sum_{l=0}^{m}\binom{m}{l}\bigg\{\mathbb{E}\Big|\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}\mathcal{G}_1^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}(h*\mathcal{G})_1^x(S_j)\Big|^l\bigg\}^{1/2}$$
$$\times\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}\mathcal{J}_{a-1}^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}(h*\mathcal{J}_{a-1})^x(S_j)\Big]^{m-l}\bigg\}^{1/2}$$

where $\mathcal{J}_{a-1}^x(S)=\sum_{k=2}^{a-1}\mathcal{G}_k^x(S\circ\theta^{n_{k-1}-n_1})$ is an additive functional on the increasing sequence $1\le n_2-n_1<\cdots<n_a-n_1$.

The approach here relies on Fourier transformation. By Parseval identity,

$$\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}\mathcal{F}_a^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{2}(h*\mathcal{F})_a^x(S_j)$$
$$=\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\big[1-|\widehat{h}(\lambda)|^2\big]\Big[\sum_{x\in\mathbb{Z}^d}e^{i\lambda\cdot x}\mathcal{F}_a^x(S_1)\Big]\overline{\Big[\sum_{x\in\mathbb{Z}^d}e^{i\lambda\cdot x}\mathcal{F}_a^x(S_1)\Big]}d\lambda$$

where

$$\widehat{h}(\lambda)=\sum_{x\in\mathbb{Z}^d}h(x)e^{i\lambda\cdot x}.$$

By independence,

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 \mathcal{F}_a^x(S_j) - \sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2 (h*\mathcal{F})_a^x(S_j)\Big]^m \qquad (6.1.18)\\
&= \frac{1}{(2\pi)^{md}}\int_{(\mathbb{R}^d)^m}\mu(d\lambda_1)\cdots\mu(d\lambda_m)\Big\vert\prod_{k=1}^m\sum_{x\in\mathbb{Z}^d} e^{i\lambda_k\cdot x}\mathcal{F}_a^x(S)\Big\vert^2
\end{aligned}
$$

where the measure $\mu(d\lambda)$ is given by

$$\mu(d\lambda) = \big[1 - \vert\widehat{h}(\lambda)\vert^2\big]d\lambda.$$

In particular, the m-th moment is non-negative. Similarly, one can see that all moments appearing in the statement of Theorem 6.1.2 are non-negative. Consequently, everything we have talked about is well defined.

Write

$$\Delta_1(\lambda) = \sum_{x\in\mathbb{Z}^d} e^{i\lambda_k\cdot x}\mathcal{G}_1^x(S), \qquad \Delta_2(\lambda) = \sum_{x\in\mathbb{Z}^d} e^{i\lambda_k\cdot x}\mathcal{J}_{a-1}^x(S)\circ\theta^{n_1}.$$

By definition of $\mathcal{J}_{a-1}^x(S)$

$$\sum_{x\in\mathbb{Z}^d} e^{i\lambda\cdot x}\mathcal{F}_a^x(S) = \Delta_1(\lambda) + \Delta_2(\lambda).$$

By triangular inequality,

$$
\begin{aligned}
&\bigg\{\int_{(\mathbb{R}^d)^m}\mu(d\lambda_1)\cdots\mu(d\lambda_m)\Big\vert\prod_{k=1}^m\sum_{x\in\mathbb{Z}^d} e^{i\lambda_k\cdot x}\mathcal{F}_a^x(S)\Big\vert^2\bigg\}^{1/2}\\
&\le \sum_{j_1,\cdots,j_m=1}^2\bigg\{\int_{(\mathbb{R}^d)^m}\mu(d\lambda_1)\cdots\mu(d\lambda_m)\Big\vert\mathbb{E}\Delta_{j_1}(\lambda_1)\cdots\Delta_{j_m}(\lambda_m)\Big\vert^2\bigg\}^{1/2}\\
&= \sum_{l=0}^m\binom{m}{l}\bigg\{\int_{(\mathbb{R}^d)^m}\mu(d\lambda_1)\cdots\mu(d\lambda_m)\Big\vert\mathbb{E}\Big(\prod_{k=1}^l\Delta_1(\lambda_k)\Big)\Big(\prod_{k=l+1}^m\Delta_2(\lambda_k)\Big)\Big\vert^2\bigg\}^{1/2}.
\end{aligned}
$$

For (6.1.17) to hold, therefore, it suffices to show that for any $0\le l\le m$,

$$
\begin{aligned}
&\int_{(\mathbb{R}^d)^m}\mu(d\lambda_1)\cdots\mu(d\lambda_m)\Big\vert\mathbb{E}\Big(\prod_{k=1}^l\Delta_1(\lambda_k)\Big)\Big(\prod_{k=l+1}^m\Delta_2(\lambda_k)\Big)\Big\vert^2 \qquad (6.1.19)\\
&\le (2\pi)^{md}\bigg\{\mathbb{E}\Big\vert\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2\mathcal{G}_1^x(S_j) - \sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2(h*\mathcal{G})_1^x(S_j)\Big\vert^l\bigg\}\\
&\times\bigg\{\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2\mathcal{J}_{a-1}^x(S_j) - \sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2(h*\mathcal{J}_{a-1})^x(S_j)\Big]^{m-l}\bigg\}.
\end{aligned}
$$

By spatial homogeneity,

$$\begin{aligned}\Delta_2(\lambda) &= \sum_{x\in\mathbb{Z}^d} e^{i\lambda_k\cdot x}\mathcal{J}_{a-1}^{x-S(n_1)}(S^{(n_1)})\\ &= e^{-i\lambda\cdot S(n_1)}\sum_{x\in\mathbb{Z}^d} e^{i\lambda_k\cdot x}\mathcal{J}_{a-1}^{x}(S^{(n_1)})\\ &= e^{-i\lambda\cdot S(n_1)}\widehat{\mathcal{J}}_{a-1}(S^{(n_1)})(\lambda)\end{aligned}$$

where $S^{(n_1}$ is the random walk

$$S^{(n_1)}(n) = S(n_1+n) - S(n_1) \qquad n = 1, 2, \cdots .$$

By increment independence,

$$\begin{aligned}&\mathbb{E}\Big(\prod_{k=1}^{l}\Delta_1(\lambda_k)\Big)\Big(\prod_{k=l+1}^{m}\Delta_2(\lambda_m)\Big)\\ &= \mathbb{E}\Big[\exp\Big\{-i\Big(\sum_{l+1}^{m}\lambda_k\Big)\cdot S(n_1)\Big\}\prod_{k=1}^{l}\Delta_1(\lambda_k)\Big]\mathbb{E}\prod_{k=l+1}^{m}\widehat{\mathcal{J}}_{a-1}(S)(\lambda_k)\\ &= \mathbb{E}\Big[\exp\Big\{-i\Big(\sum_{l+1}^{m}\lambda_k\Big)\cdot S(n_1)\Big\}\prod_{k=1}^{l}\widehat{\mathcal{G}}_1(S)(\lambda_k)\Big]\mathbb{E}\prod_{k=l+1}^{m}\widehat{\mathcal{J}}_{a-1}(S)(\lambda_k).\end{aligned}$$

So we have the equality

$$\begin{aligned}&\Big|\mathbb{E}\Big(\prod_{k=1}^{l}\Delta_1(\lambda_k)\Big)\Big(\prod_{k=l+1}^{m}\Delta_2(\lambda_m)\Big)\Big|^2\\ &= \Big|\mathbb{E}\prod_{k=l+1}^{m}\widehat{\mathcal{J}}_{a-1}(S)(\lambda_k)\Big|^2\mathbb{E}\Big\{\exp\Big\{i\Big(\sum_{l+1}^{m}\lambda_k\Big)\cdot\big(S_2(n_1)-S_1(n_1)\big)\Big\}\\ &\times\prod_{k=1}^{l}\widehat{\mathcal{G}}_1(S_1)(\lambda_k)\overline{\widehat{\mathcal{G}}_1(S_2)(\lambda_k)}\Big\}.\end{aligned}$$

Fix $(\lambda_{l+1},\cdots,\lambda_m)$ for a moment and take integration against $\mu(d\lambda_1)\cdots\mu(d\lambda_l)$ on the both side.

$$\begin{aligned}&\int_{([-\pi,\pi]^d)^l}\mu(d\lambda_1)\cdots\mu(d\lambda_l)\Big|\mathbb{E}\Big(\prod_{k=1}^{l}\Delta_1(\lambda_k)\Big)\Big(\prod_{k=l+1}^{m}\Delta_2(\lambda_m)\Big)\Big|^2\\ &= \Big|\mathbb{E}\prod_{k=l+1}^{m}\widehat{\mathcal{J}}_{a-1}(S)(\lambda_k)\Big|^2\mathbb{E}\Big\{\exp\Big\{i\Big(\sum_{l+1}^{m}\lambda_k\Big)\cdot\big(S_2(n_1)-S_1(n_1)\big)\Big\}\\ &\times\Big[\int_{[-\pi,\pi]^d}\widehat{\mathcal{G}}_1(S_1)(\lambda)\overline{\widehat{\mathcal{G}}_1(S_2)(\lambda)}d\lambda\Big]^l\Big\}.\end{aligned}$$

By Parseval identity,

$$\int_{[-\pi,\pi]^d}\widehat{\mathcal{G}}_1(S_1)(\lambda)\overline{\widehat{\mathcal{G}}_1(S_2)(\lambda)}d\lambda=(2\pi)^{ld}\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2\mathcal{G}_1^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2(h*\mathcal{G})_1^x(S_j).$$

Using the fact that

$$\Big|\exp\Big\{i(\lambda_{b+1}+\cdots+\lambda_m)\cdot\big(\widetilde{S}(n_1)-S(n_1)\big)\Big\}\Big|=1$$

we conclude that

$$\begin{aligned}&\int_{([-\pi,\pi]^d)^m}d\lambda_1\cdots d\lambda_m\Big(\prod_{k=1}^m\big[1-\widehat{h}(\lambda_k)\big]\Big)\Big|\mathbb{E}\prod_{k=b+1}^m D_2(\lambda_k)\Big|^2\\&\le(2\pi)^{ld}\mathbb{E}\Big|\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2\mathcal{G}_1^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2(h*\mathcal{G})_1^x(S_j)\Big|^l\Big|\mathbb{E}\prod_{k=l+1}^m\widehat{\mathcal{J}}_{a-1}(S)(\lambda_k)\Big|^2.\end{aligned}$$

Take integration against $\mu(d\lambda_{l+1})\cdots\mu(d\lambda_m)$ on the both sides. The requested (6.1.19) follows from the equality

$$\begin{aligned}&\int_{([-\pi,\pi]^d)^{m-l}}\Big|\mathbb{E}\prod_{k=l+1}^m\widehat{\mathcal{J}}_{a-1}(\lambda_k)\Big|^2\mu(d\lambda_{l+1})\cdots\mu(d\lambda_m)\\&=(2\pi)^{(m-l)d}\mathbb{E}\Big|\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2\mathcal{J}_{a-1}^x(S_j)-\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2(h*\mathcal{J}_{a-1})^x(S_j)\Big|^{m-l}.\end{aligned}$$

As for the proof of (6.1.14), observe that correspondent to (6.1.18), we have

$$\begin{aligned}&\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^2\Big(\mathcal{F}_k^x(S_j)-(h*\mathcal{F})_k^x(S_j)\Big)\Big]^m\\&=\frac{1}{(2\pi)^{md}}\int_{(\mathbb{R}^d)^m}\nu(d\lambda_1)\cdots\nu(d\lambda_m)\Big|\prod_{k=1}^m\sum_{x\in\mathbb{Z}^d}e^{i\lambda_k\cdot x}\mathcal{F}_k^x(S)\Big|^2\end{aligned}$$

where the measure $\nu(d\lambda)$ is given by

$$\nu(d\lambda)=|1-\widehat{h}(\lambda)|^2d\lambda.$$

The rest follows an essentially same argument. □

The additivity becomes an important issue in order to apply Theorem 6.1.2. As a consequence in the special models discussed previously, Theorem 6.1.2 applies to the case when $\mathcal{F}_k^x=l(n_k,x)$, but not to the case when $\mathcal{F}_k^x=1_{\{x\in S[1,n_k]\}}$. The problem existing in the later case creates some difficulties in approximating J_n by its smoothed versions. That is a major reason why we introduce the additive version defined in (6.1.6).

6.2 Integrability of I_n and J_n

In this section, we apply Theorem 6.1.1 to establish the maximal integrability for the mutual intersection local time

$$I_n = \sum_{k_1,\cdots,k_p=1}^{n} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}}$$

and intersection of independent ranges

$$J_n = \#\big\{S_1[1,n]\times\cdots\times S_p[1,n]\big\}$$

run by the independent, identically distributed symmetric random walks $S_1(k),\cdots,S_p(k)$. We assume $S_1(k),\cdots,S_p(k)$ are square integrable with covariance matrix Γ and the smallest sub-group of $\mathbb{Z}^d$ that supports the random walks is $\mathbb{Z}^d$ itself. Throughout the section, we assume that $p(d-2)<d$.

Theorem 6.2.1 *Under $p(d-2)<d$, there is a constant $C>0$ such that*

$$\mathbb{E}I_n^m \le (m!)^{\frac{d(p-1)}{2}} C^m n^{\frac{2p-d(p-1)}{2}} \qquad m,n=1,2,\cdots. \tag{6.2.1}$$

Consequently, there is a $\theta>0$ such that

$$\sup_{n\ge 1}\mathbb{E}\exp\Big\{\theta n^{-\frac{2p-d(p-1)}{d(p-1)}} I_n^{\frac{2}{d(p-1)}}\Big\}<\infty. \tag{6.2.2}$$

For the combinations of d and p with constraint $p(d-2)<d$,

(1) When $d=1$ and $p\ge 1$ (That's right, we include the case $p=1$ here)

$$\mathbb{E}J_n^m \le (m!)^{1/2}C^m n^{m/2} \tag{6.2.3}$$

Consequently, there is a $\theta>0$ such that

$$\sup_{n\ge 1}\mathbb{E}\exp\Big\{\theta n^{-1}J_n^2\Big\}<\infty. \tag{6.2.4}$$

(2) When $d=2$ and $p\ge 2$,

$$\mathbb{E}J_n^m \le (m!)^{p-1}C^m\Big(\frac{n}{(\log n)^p}\Big)^m \tag{6.2.5}$$

Consequently, there is a $\theta>0$ such that

$$\sup_{n\ge 1}\mathbb{E}\exp\Big\{\theta n^{-\frac{1}{p-1}}(\log n)^{\frac{p}{p-1}}J_n^{\frac{1}{p-1}}\Big\}<\infty. \tag{6.2.6}$$

(3) When $d = 3$ and $p = 2$,

$$\mathbb{E}J_n^m \le (m!)^{3/2} C^m n^{m/2} \tag{6.2.7}$$

Consequently, there is a $\theta > 0$ such that

$$\sup_{n\ge 1} \mathbb{E}\exp\Big\{\theta n^{-1/3} J_n^{2/3}\Big\} < \infty. \tag{6.2.8}$$

As the first step of the proof, we establish the following weak version of Theorem 6.1.1.

Lemma 6.2.2 *Under $p(d-2) < d$, there is a constant $C > 0$ such that*

$$\mathbb{E}I_n^m \le (m!)^p C^m n^{\frac{2p-d(p-1)}{2}} \qquad m, n = 1, 2, \cdots . \tag{6.2.9}$$

For the combinations of d and p with constraint $p(d-2) < d$, the moments of J_n yield the following bounds.

(1) When $d = 1$ and $p \ge 2$,

$$\mathbb{E}J_n^m \le (m!)^p C^m n^{m/2}. \tag{6.2.10}$$

(2) When $d = 2$ and $p \ge 2$,

$$\mathbb{E}J_n^m \le (m!)^p C^m \Big(\frac{n}{(\log n)^p}\Big)^m. \tag{6.2.11}$$

(3) When $d = 3$ and $p = 2$,

$$\mathbb{E}J_n^m \le (m!)^2 C^m n^{m/2}. \tag{6.2.12}$$

Proof. For similarity we estimate the moments of J_n. The discussion on the weak convergence in the last chapter gives the asymptotic orders of J_n in the above three situations. By that it is sufficient to prove that

$$\mathbb{E}J_n^m \le (m!)^p C^m \big(p + \mathbb{E}J_n\big)^m \qquad m, n = 1, 2, \cdots . \tag{6.2.13}$$

By the representation

$$J_n = \sum_{x\in\mathbb{Z}^d} \prod_{j=1}^p 1_{\{T_x^{(j)} \le n\}}$$

where $T_x^{(j)} = \inf\{k \ge 1;\quad S_j(k) = x\}$, and by Independence

$$\begin{aligned}\mathbb{E}J_n^m &= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{P}\{T_{x_1}\le n,\cdots,T_{x_m}\le n\}\Big]^p\\ &= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\bigg[\sum_{\sigma\in\Sigma_m}\mathbb{P}\big\{T_{x_{\sigma(1)}}\le\cdots\le T_{x_{\sigma(m)}}\le n\big\}\bigg]^p.\end{aligned}$$

Notice that

$$\begin{aligned}&\mathbb{P}\big\{T_{x_{\sigma(1)}}\le\cdots\le T_{x_{\sigma(m)}}\le n\big\}\\ &=\sum_{1\le j\le k\le n}\mathbb{P}\big\{T_{x_{\sigma(1)}}\le\cdots\le T_{x_{\sigma(m-1)}},\ T_{x_{\sigma(m-1)}}=j,\ T_{x_{\sigma(m)}}=k\big\}\\ &\le\sum_{1\le j\le k\le n}\mathbb{P}\big\{T_{x_{\sigma(1)}}\le\cdots\le T_{x_{\sigma(m-1)}},\ T_{x_{\sigma(m-1)}}=j,\\ &\qquad\qquad S(j+1)\ne x_{\sigma(m)},\cdots,S(k-1)\ne x_{\sigma(m)},\ S(k)=x_{\sigma(m)}\big\}\\ &=\sum_{1\le j\le k\le n}\mathbb{P}\big\{T_{x_{\sigma(1)}}\le\cdots\le T_{x_{\sigma(m-1)}},\ T_{x_{\sigma(m-1)}}=j\big\}\mathbb{P}\big\{\widetilde{T}_{x_{\sigma(m)}-x_{\sigma(m-1)}}=k-j\big\}\end{aligned}$$

where (note that the time starts at 0)

$$\widetilde{T}_x = \inf\{k\ge 0;\quad S(k)=x\}\qquad x\in\mathbb{Z}^d.$$

Thus,

$$\begin{aligned}&\mathbb{P}\big\{T_{x_{\sigma(1)}}\le\cdots\le T_{x_{\sigma(m)}}\le n\big\}\\ &\le\bigg\{\sum_{j=1}^n\mathbb{P}\big\{T_{x_{\sigma(1)}}\le\cdots\le T_{x_{\sigma(m-1)}},\ T_{x_{\sigma(m-1)}}=j\big\}\bigg\}\\ &\qquad\times\bigg\{\sum_{k=0}^n\mathbb{P}\big\{\widetilde{T}_{x_{\sigma(m)}-x_{\sigma(m-1)}}=k\big\}\bigg\}\\ &=\mathbb{P}\big\{T_{x_{\sigma(1)}}\le\cdots\le T_{x_{\sigma(m-1)}}\le n\big\}\mathbb{P}\big\{\widetilde{T}_{x_{\sigma(m)}-x_{\sigma(m-1)}}\le n\big\}.\end{aligned}$$

Repeating above procedure,

$$\mathbb{P}\big\{T_{x_{\sigma(1)}}\le\cdots\le T_{x_{\sigma(m)}}\le n\big\}\le\prod_{k=1}^m\mathbb{P}\big\{\widetilde{T}_{x_{\sigma(k)}-x_{\sigma(k-1)}}\le n\big\}.$$

where we follow the convention $x_{\sigma(0)}=0$.

Therefore, by Jensen's inequality

$$\begin{aligned}
\mathbb{E}J_n^m &\le \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m \mathbb{P}\big\{\widetilde{T}_{x_{\sigma(k)}-x_{\sigma(k-1)}}\le n\big\}\Big]^p\\
&\le (m!)^{p-1}\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m\Big(\mathbb{P}\big\{\widetilde{T}_{x_{\sigma(k)}-x_{\sigma(k-1)}}\le n\big\}\Big)^p\\
&= (m!)^p\Big\{\sum_{x\in\mathbb{Z}^d}\Big(\mathbb{P}\big\{\widetilde{T}_x\le n\big\}\Big)^p\Big\}^m.
\end{aligned}$$

Finally, (6.2.13) follows from the fact that

$$\sum_{x\in\mathbb{Z}^d}\Big(\mathbb{P}\big\{\widetilde{T}_x\le n\big\}\Big)^p = \mathbb{E}\Big[\#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\}\Big\}\le p+\mathbb{E}J_n.$$

□

Proof of Theorem 6.2.1. By Taylor expansion, the exponential integrabilities stated in (6.2.2), (6.2.4), (6.2.6) and (6.2.8) are the consequences of the bounds given in (6.2.1), (6.2.3), (6.2.5) and (6.2.7), respectively. Among these bounds we only prove (6.2.1) for the reason of similarity. A striking feature in our argument is that we improve the bound (6.2.9) based on (6.2.9) itself.

We first deal with the case $m\le n$. By (6.2.9) and by the fact $m\le n$, there is a constant $C>0$ (which is allows to be different from place to place in our proof) such that

$$\mathbb{E}I_{[n/m]+1}^{k_i}\le C^{k_i}(k_i!)^p\Big(\frac{n}{m}\Big)^{\frac{2p-d(p-1)}{2}k_i}\qquad i=1,\cdots,m.$$

Taking $a=m$ and $n_k-n_{k-1}=[n/m]+1$ in the moment inequality (6.1.7),

$$\begin{aligned}
\Big(\mathbb{E}I_n^m\Big)^{1/p} &\le \sum_{\substack{k_1+\cdots+k_m=m\\ k_1,\cdots,k_m\ge 0}}\frac{m!}{k_1!\cdots k_m!}C^{m/p}k_1!\cdots k_m!\Big(\frac{n}{m}\Big)^{\frac{2p-d(p-1)}{2p}m}\\
&= m!C^{m/p}(m^m)^{-\frac{2p-d(p-1)}{2p}}n^{\frac{2p-d(p-1)}{2p}m}\binom{2m-1}{m}.
\end{aligned}$$

Therefore, (6.2.1) follows from the easy bounds $m^m\ge m!$ and

$$\binom{2m-1}{m}\le\sum_{k=0}^{2m-1}\binom{2m-1}{k}=2^{2m-1}.$$

As for the case $m > n$, by the fact that $I_n \le n^p$ we have the trivial bound

$$\mathbb{E}I_n^m \le n^{pm} \le m^{\frac{d(p-1)}{2}m} n^{\frac{2p-d(p-1)}{2}m} \le C^m (m!)^{\frac{d(p-1)}{2}} n^{\frac{2p-d(p-1)}{2}m}$$

where the last step follows from Stirling formula. □

Remark 6.2.3 *The integrability established for I_n and J_n are best possible in the sense that there is a $\theta_0 > 0$ such that left hand sides of (6.2.2), (6.2.4), (6.2.6) and (6.2.8) become infinity if $\theta > \theta_0$. For example, the weak convergence stated in Theorem 5.3.2 and the critical integrability stated in Remark 3.3.5 imply that*

$$\lim_{n\to\infty} \mathbb{E}\exp\left\{\theta n^{-\frac{2p-d(p-1)}{d(p-1)}} I_n^{\frac{2}{d(p-1)}}\right\} = \infty$$

for any $\theta > \dfrac{p}{2}\det(\Gamma)^{\frac{p-1}{2}}\kappa(d,p)^{-\frac{4p}{d(p-1)}}$.

6.3 Integrability of Q_n and R_n in low dimensions

We mainly consider the case $d = 2$ (with a few exceptions where $d = 3$ or higher), in which Q_n and R_n concentrate around their means. Our first result is on the exponential integrability of R_n.

Theorem 6.3.1 *When $d = 2$,*

$$\sup_{n\ge 1} \mathbb{E}\exp\left\{\theta \frac{\log n}{n} R_n\right\} < \infty \qquad \forall \theta > 0. \tag{6.3.1}$$

When $d \ge 3$,

$$\sup_{n\ge 1} \mathbb{E}\exp\left\{\frac{\theta}{n} R_n\right\} < \infty \qquad \forall \theta > 0. \tag{6.3.2}$$

Proof. We apply Theorem 1.3.3 here. First notice that the process R_n is sub-additive in the sense that for any integers $m, n \ge 1$,

$$R_{n+m} \le R_n + \widetilde{R}_m \tag{6.3.3}$$

where $\widetilde{R}_m = \#\{S_{n+1}, \cdots, S_{n+m}\}$ is independent of $\{S(1), \cdots, S(n)\}$ and has a distribution same as R_m.

Second, by the known fact that $\mathbb{E}R_n = O\Big(\frac{n}{\log n}\Big)$ when $d = 2$ and $\mathbb{E}R_n = O(n)$ when $d \ge 3$, and by the fact that the sub-additive sequence R_n is non-decreasing, so the assumption on stochastic boundedness posted in Theorem 1.3.3 holds with $c_n = \frac{n}{\log n}$ when $d = 2$ and $c_n = n$ when $d \ge 3$. $\square$

Due to the concentration phenomena in the case $d \ge 2$, we are more interested in the integrability of $Q_n - \mathbb{E}Q_n$ and $R_n - \mathbb{E}R_n$. Here is the main result of this section.

Theorem 6.3.2 . *There is a $\theta > 0$ such that*

(1). As $d = 2$,

$$\sup_{n\ge 1} \mathbb{E}\exp\Big\{\theta \frac{(\log n)^2}{n}\big|R_n - \mathbb{E}R_n\big|\Big\} < \infty, \tag{6.3.4}$$

$$\sup_{n\ge 1} \mathbb{E}\exp\Big\{\theta \frac{1}{n}\big|Q_n - \mathbb{E}Q_n\big|\Big\} < \infty. \tag{6.3.5}$$

(2). As $d = 3$,

$$\sup_{n\ge 1} \mathbb{E}\exp\Big\{\theta \frac{1}{n^{2/3}}\big|R_n - \mathbb{E}R_n\big|\Big\} < \infty. \tag{6.3.6}$$

Proof. We first consider the case $d = 2$ and prove (6.3.4) with n being replaced by 2^n. To this end, all we need is to show that there is a $\theta > 0$ such that

$$\sup_{n\ge 1} \mathbb{E}\exp\Big\{\theta \frac{(\log 2^n)^2}{2^n}\big(R_n - \mathbb{E}R_n\big)\Big\}, < \infty \tag{6.3.7}$$

$$\sup_{n\ge 1} \mathbb{E}\exp\Big\{-\theta \frac{(\log 2^n)^2}{2^n}\big(R_n - \mathbb{E}R_n\big)\Big\} < \infty. \tag{6.3.8}$$

We only prove (6.3.7) for similarity.

The proof is based on the procedure of triangular approximation used for (2.4.7) in Theorem 2.4.2 and the reader may compare the decomposition (6.3.9) with Figure 2.1 in Section 2.4 for some geometric insight.

Let

$$N = [2(\log 2)^{-1}\log n].$$

It is easy to see that $2^N \sim n^2$. Set

$$\beta_k = \#\{S((k-1)2^{n-N}, k2^{n-N}]\}$$

$$\alpha_{j,k} = \#\Big\{S\big((2k-2)2^{n-j}, (2k-1)2^{n-j}\big] \cap S\big((2k-1)2^{n-j}, (2k)2^{n-j}\big]\Big\}.$$

By inclusion-exclusion rule we have the decomposition

$$R_{2^n} - \mathbb{E}R_{2^n} = \sum_{k=1}^{2^N} \overline{\beta}_k - \sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{\alpha}_{j,k} \tag{6.3.9}$$

where $\overline{\beta}_k = \beta_k - \mathbb{E}\beta_k$ and $\overline{\alpha}_{j,k} = \alpha_{j,k} - \mathbb{E}\alpha_{j,k}$. By Theorem 6.3.1,

$$\sup_n \mathbb{E}\exp\Big\{\lambda \frac{\log 2^{n-N}}{2^{n-N}} |\overline{\beta}_1|\Big\} < \infty \qquad \forall \lambda > 0.$$

By Taylor expansion, there is a $C > 0$ and $\lambda_0 > 0$ independent of n such that

$$\mathbb{E}\exp\Big\{\lambda \frac{\log 2^{n-N}}{2^{n-N}} \overline{\beta}_1\Big\} \le \exp\Big\{C\lambda^2\Big\} \qquad \forall 0 \le \lambda \le \lambda_0.$$

Notice that $\overline{\beta}_1, \cdots, \overline{\beta}_{2^N}$ is an i.i.d. sequence with $\mathbb{E}\overline{\beta}_1 = 0$. Thus,

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\theta 2^{-N/2} \frac{\log 2^{n-N}}{2^{n-N}} \sum_{k=1}^{2^N} \overline{\beta}_k\Big\} \\
&= \bigg(\mathbb{E}\exp\Big\{\theta 2^{-N/2} \frac{\log 2^{n-N}}{2^{n-N}} \overline{\beta}_k\Big\}\bigg)^{2^N} \le \exp\Big\{C\theta^2\Big\}.
\end{aligned}$$

Consequently, there is a $\theta > 0$ such that

$$\sup_n \mathbb{E}\exp\Big\{\theta 2^{-N/2} \frac{\log 2^{n-N}}{2^{n-N}} \sum_{k=1}^{2^N} \overline{\beta}_k\Big\} < \infty.$$

By the choice of N one can see that there is a $c > 0$ independent of n such that

$$2^{-N/2} \frac{\log 2^{n-N}}{2^{n-N}} \ge c \frac{(\log 2^n)^2}{2^n}.$$

Therefore, there is some $\theta > 0$ such that

$$\sup_n \mathbb{E}\exp\Big\{\theta \frac{(\log 2^n)^2}{2^n} \sum_{k=1}^{2^N} \overline{\beta}_k\Big\} < \infty. \tag{6.3.10}$$

We need to show that for some $\theta > 0$,

$$\sup_n \mathbb{E}\exp\left\{-\theta\frac{(\log 2^n)^2}{2^n}\sum_{j=1}^N\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\right\} < \infty. \tag{6.3.11}$$

Set

$$\tilde{J}_n = \#\{S[1,n]\cap\widetilde{S}[1,n]\} \qquad n = 1, 2, \cdots,$$

where $\widetilde{S}$ is an independent copy of the random walk S. In our notation, for each $1 \le j \le N$, $\{\overline{\alpha}_{j,1}, \cdots, \overline{\alpha}_{j,2^{j-1}}\}$ is an i.i.d. sequence with the same distribution as $\tilde{J}_{2^{n-j}}$. By (6.2.2) in Theorem 6.2.1 (with $d = p = 2$), there is a $\delta > 0$ such that

$$\sup_n\sup_{j\le N}\mathbb{E}\exp\left\{\delta\frac{(\log 2^{n-j})^2}{2^{n-j}}\big|\overline{\alpha}_{j,1}\big|\right\} < \infty.$$

By the argument used for (6.3.10), there is a $\bar{\theta} > 0$ such that

$$\sup_n\sup_{j\le N}\mathbb{E}\exp\left\{-\bar{\theta}2^{-j/2}\frac{(\log 2^n)^2}{2^{n-j}}\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\right\} < \infty.$$

Hence for some $\theta > 0$

$$C(\theta) \equiv \sup_n\sup_{j\le N}\mathbb{E}\exp\left\{-\theta 2^{j/2}\frac{(\log 2^n)^2}{2^n}\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\right\} < \infty.$$

Write

$$\lambda_N = \prod_{j=1}^N\left(1 - 2^{-j/2}\right) \quad\text{and}\quad \lambda_\infty = \prod_{j=1}^\infty\left(1 - 2^{-j/2}\right).$$

Using Hölder's inequality with $1/p = 1 - 2^{-N/2}$, $1/q = 2^{-N/2}$ we have

$$\begin{aligned}
&\mathbb{E}\exp\left\{-\lambda_N\theta\frac{(\log 2^n)^2}{2^n}\sum_{j=1}^N\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\right\}\\
&\le\left(\mathbb{E}\exp\left\{-\lambda_{N-1}\theta\frac{(\log 2^n)^2}{2^n}\sum_{j=1}^{N-1}\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\right\}\right)^{1-2^{-N/2}}\\
&\qquad\times\left(\mathbb{E}\exp\left\{-\lambda_N 2^{N/2}\frac{(\log 2^n)^2}{2^n}\sum_{k=1}^{2^{N-1}}\overline{\alpha}_{N,k}\right\}\right)^{2^{-N/2}}\\
&\le\mathbb{E}\exp\left\{-\lambda_{N-1}\theta\frac{(\log 2^n)^2}{2^n}\sum_{j=1}^{N-1}\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\right\}\cdot C(\theta)^{2^{-N/2}}
\end{aligned}$$

where the last step partially follows from the fact (by Jensen inequality) that

$$\mathbb{E}\exp\left\{-\lambda_{N-1}\theta\frac{(\log 2^n)^2}{2^n}\sum_{j=1}^{N-1}\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\right\}\geq 1.$$

Repeating this procedure,

$$\begin{aligned}&\mathbb{E}\exp\left\{\lambda_N\theta\frac{(\log 2^n)^2}{2^n}\Big|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\Big|\right\}\\ &\leq C(\theta)^{2^{-1/2}+\cdots+2^{-N/2}}\leq C(\theta)^{2^{-1/2}(1-2^{-1/2})^{-1}}.\end{aligned}$$

Thus,

$$\sup_n\mathbb{E}\exp\left\{\lambda_\infty\theta\frac{(\log 2^n)^2}{2^n}\Big|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}}\overline{\alpha}_{j,k}\Big|\right\}\leq C(\theta)^{2^{-1/2}(1-2^{-1/2})^{-1}}.$$

So we have (6.3.11).

Hence, (6.3.7) follows from (6.3.9), (6.3.10) and (6.3.11). By (6.3.7) and (6.3.8), there is $\theta>0$ such that

$$\sup_{n\geq 1}\mathbb{E}\exp\left\{\theta\frac{(\log 2^n)^2}{2^n}\big|R_n-\mathbb{E}R_n\big|\right\}<\infty. \tag{6.3.12}$$

We now extend (6.3.12) to (6.3.4). Given an integer $n\geq 2$, we have the following unique binary representation:

$$n=2^{m_1}+2^{m_2}+\cdots+2^{m_l}$$

where $m_1>m_2>\cdots m_l\geq 0$ are integers. Write

$$n_0=0\quad\text{and}\quad n_i=2^{m_1}+\cdots+2^{m_i}\qquad i=1,\cdots,l.$$

By inclusion-exclusion,

$$\begin{aligned}R_n&=\sum_{i=1}^{l}\#\big\{S(n_{i-1},n_i]\big\}-\sum_{i=1}^{l-1}\#\big\{S(n_{i-1},n_i]\cap S(n_i,n]\big\} \\ &=\sum_{i=1}^{l}B_i-\sum_{i=1}^{l-1}A_i\qquad\text{(say)}.\end{aligned}\tag{6.3.13}$$

Write

$$\sum_{i=1}^{l}B_i=\sum_i{}'B_i+\sum_i{}''B_i$$

where $\sum_i{}'$ is the summation over i with $2^{m_i} \ge \sqrt{n}$ and $\sum_i{}''$ is the summation over i with $2^{m_i} < \sqrt{n}$. We also define the products $\prod_i{}'$ and $\prod_i{}''$ in a similar manner. Notice that

$$B_i \stackrel{d}{=} R_{2^{m_i}} \qquad i = 1, \cdots, l.$$

By Hölder inequality,

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}\Big|\sum_i{}'(B_i - \mathbb{E}B_i)\Big|\Big\} \\
&\le \prod_i{}'\Big(\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}2^{-m_i}\Big(\sum_j{}'2^{m_j}\Big)\big|R_{2^{m_i}} - \mathbb{E}R_{2^{m_i}}\big|\Big\}\Big)^{2^{m_i}\left(\sum_j{}'2^{m_j}\right)^{-1}} \\
&\le \prod_i{}'\Big(\mathbb{E}\exp\Big\{4\theta\frac{(\log 2^{m_i})^2}{2^{m_i}}\big|R_{2^{m_i}} - \mathbb{E}R_{2^{m_i}}\big|\Big\}\Big)^{2^{m_i}\left(\sum_j{}'2^{m_j}\right)^{-1}} \\
&\le \sup_m \mathbb{E}\exp\Big\{4\theta\frac{(\log 2^m)^2}{2^m}\big|R_n - \mathbb{E}R_n\big|\Big\}.
\end{aligned}$$

By (6.3.12), the right hand side is finite for some $\theta > 0$.

Assume that the set $\{1 \le i \le l;\ 2^{m_i} < \sqrt{n}\}$ is non-empty. We have

$$\sum_i{}''2^{m_i} \le 2\sqrt{n}.$$

So we have

$$\frac{(\log n)^2}{n} \le \frac{1}{\sqrt{n}} \le 2\Big(\sum_i{}''2^{m_i}\Big)^{-1}.$$

Hence

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}\Big|\sum_i{}''(B_i - \mathbb{E}B_i)\Big|\Big\} \\
&\le \prod_i{}''\Big(\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}2^{-m_i}\Big(\sum_j{}''2^{m_j}\Big)\big|R_{2^{m_i}} - \mathbb{E}R_{2^{m_i}}\big|\Big\}\Big)^{2^{m_i}\left(\sum_j{}''2^{m_j}\right)^{-1}} \\
&\le \prod_i{}''\Big(\mathbb{E}\exp\Big\{2\theta\frac{1}{2^{m_i}}\big|R_{2^{m_i}} - \mathbb{E}R_{2^{m_i}}\big|\Big\}\Big)^{2^{m_i}\left(\sum_j{}''2^{m_j}\right)^{-1}} \\
&\le \sup_m \mathbb{E}\exp\Big\{2\theta\frac{1}{2^m}\big|R_{2^m} - \mathbb{E}R_{2^m}\big|\Big\}.
\end{aligned}$$

By Cauchy-Schwartz inequality, therefore, there exists $\theta > 0$ such that

$$\sup_{n\geq 1}\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}\Big|\sum_{i=1}^{l}(B_i-\mathbb{E}B_i)\Big|\Big\}<\infty. \tag{6.3.14}$$

By the fact that

$$n-n_i=2^{m_{i+1}}+\cdots+2^{m_l}\leq 2^{m_i}$$

we have

$$A_i\overset{d}{=}\#\big\{S[1,2^{m_i}]\cap S'[1,n-n_i]\big\}\leq J_{2^{m_i}}. \tag{6.3.15}$$

Notice that ((6.2.11) in Lemma 6.2.2, $m=1$, $p=2$)

$$\mathbb{E}J_n=O\Big(\frac{n}{(\log n)^2}\Big)\qquad (n\to\infty).$$

There is a constant $C>0$ independent of n such that

$$\begin{aligned}\sum_{i=1}^{l-1}\mathbb{E}A_i&\leq\sum_{i=1}^{l-1}\mathbb{E}J_{2^{m_i}}\leq C\sum_{i=1}^{l}\frac{2^{m_i}}{m_i^2}\\&\leq C\sum_{m_i<l/2}\frac{2^{m_i}}{m_i^2}+C\sum_{m_i\geq l/2}^{l}\frac{2^{m_i}}{m_i^2}\\&\leq C2^{l/2}+C\frac{n}{(\log n)^2}\leq C\frac{n}{(\log n)^2}.\end{aligned}$$

By (6.3.13) and (6.3.14), therefore, we will have (6.3.4) if we can prove

$$\sup_{n}\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}\sum_{i=1}^{l-1}A_i\Big\}<\infty.$$

Thanks to (6.2.6) in Theorem 6.2.1 (with $p=2$) and (6.3.15), this requested conclusion follows from an argument similar to the one used for (6.3.14).

The proof of (6.3.5) is similar and easier (for having no "$\log n$" part) Indeed, applying the decomposition

$$Q_{2^n}=2^n+\sum_{j=1}^{n}\sum_{k=1}^{2^{j-1}}\sum_{i=(2k-2)2^{n-j}+1}^{(2k-1)2^{n-j}}\sum_{l=(2k-1)2^{n-j}+1}^{(2k)2^{n-j}}1_{\{S(i)=S(l)\}} \tag{6.3.16}$$

and (6.2.2) (with $d=p=2$) in a similar way gives that for some $\theta>0$,

$$\sup_{n\geq 1}\mathbb{E}\exp\Big\{2^{-n}\theta\Big|Q_{2^n}-\mathbb{E}Q_{2^n}\Big|\Big\}<\infty.$$

The argument of binary expansion used above extends the above conclusion into (6.3.5).

To prove (6.3.6) we first claim that as $d = 3$ and $p = 2$,

$$\sup_{n\geq 1} \mathbb{E}\exp\Big\{\theta\frac{1}{n^{2/3}}J_n\Big\} < \infty \tag{6.3.17}$$

for some $\theta > 0$.

Indeed, by the fact that $J_n \leq n$,

$$\mathbb{E}\exp\Big\{\theta\frac{1}{n^{2/3}}J_n\Big\} \leq \mathbb{E}\exp\Big\{\theta\frac{1}{n^{1/3}}J_n^{2/3}\Big\}.$$

Hence, (6.3.17) follows from (6.2.8).

Therefore, (6.3.6) can be proved in almost same way as (6.3.4) is proved, except that we use (6.3.17) instead of (6.2.6) in the proof.[2] $\square$

By the fact (Theorem 5.4.3) that $Q_n - \mathbb{E}Q_n$ and $\mathbb{E}R_n - R_n$ converge, if properly normalized, to the remormalized self-intersection local time $\gamma\big([0,1]^2_<\big)$, and by the fact (Theorem 4.3.1 and Theorem 4.3.2) that $\gamma\big([0,1]^2_<\big)$ has asymmetric tails, we expect that $\mathbb{E}Q_n - Q_n$ and $R_n - \mathbb{E}R_n$ have tails lighter than the tails of $Q_n - \mathbb{E}Q_n$ and $\mathbb{E}R_n - R_n$ do. Indeed, we have

Theorem 6.3.3 . *The following statements hold for every $\theta > 0$:*

(1). As $d = 2$,

$$\sup_{n\geq 1} \mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}\big(R_n - \mathbb{E}R_n\big)\Big\} < \infty, \tag{6.3.18}$$

$$\sup_{n\geq 1} \mathbb{E}\exp\Big\{-\theta\frac{1}{n}\big(Q_n - \mathbb{E}Q_n\big)\Big\} < \infty. \tag{6.3.19}$$

(2). As $d = 3$,

$$\sup_{n\geq 1} \mathbb{E}\exp\Big\{\theta\frac{1}{n^{2/3}}\big(R_n - \mathbb{E}R_n\big)\Big\} < \infty. \tag{6.3.20}$$

Proof. Due to similarity, we only consider (6.3.18). By (6.3.4) in Theorem 6.3.2, there is a $\theta_o > 0$ such that(6.3.18) holds for $\theta \leq \theta_o$.

[2]A fact crucial to our argument given in the above proof is that the normalizing exponent, which is 1 in (6.3.4), (6.3.5) and is 2/3 in (6.3.6), is always greater than 1/2.

Assume that $\theta > \theta_0$. Take $a > 0$ such that $\theta < a\theta_o$. Write $n = n_1 + \cdots + n_a$ such that $n_1 \sim a^{-1}n$.

$$R_n \le \sum_{k=1}^{a} \widetilde{R}_{n_k}$$

where $\widetilde{R}_{n_k}$ $(k = 1, \cdots, a)$ are independent and $\widetilde{R}_{n_k} \overset{d}{=} R_{n_k}$. In addition, write $m_0 = 1$ and $m_k = n_1 + \cdots + n_k$.

$$\mathbb{E}R_n \ge \sum_{k=1}^{a} \mathbb{E}R_{n_k} - \sum_{k=1}^{a-1} \#\big\{S[m_k, m_{k+1}] \cap S[m_k, n]\big\}.$$

It is easy to see that the second term is bounded by a (universal) constant multiple of $n/(\log n)^2$. Consequently,

$$\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}\big(R_n - \mathbb{E}R_n\big)\Big\} \le C\prod_{k=1}^{a}\mathbb{E}\exp\Big\{\theta\frac{(\log n)^2}{n}\big(R_{n_k} - \mathbb{E}R_{n_k}\big)\Big\}$$

where $C > 0$ is independent of n. By the choice of n_k, the right hand side is bounded uniformly in n. $\square$

We now comment on the integrabilities established by far. By the weak laws stated in Theorem 5.4.2 and Theorem 5.4.3, and by the critical integrability given in (4.3.3), the integrabilities stated in the case $d = 2$ ((6.3.4) and (6.3.5)) are best possible in the sense that they are no longer hold if θ is too large. More precisely, we can see that

$$\lim_{n\to\infty} \mathbb{E}\exp\Big\{-\theta\frac{(\log n)^2}{n}(R_n - \mathbb{E}R_n)\Big\} = \infty \tag{6.3.21}$$

if $\theta > (2\pi)^{-2}\det(\Gamma)^{-1/2}\kappa(2,2)^{-4}$; and that

$$\lim_{n\to\infty} \mathbb{E}\exp\Big\{\theta\frac{1}{n}(Q_n - \mathbb{E}Q_n)\Big\} = \infty \tag{6.3.22}$$

if $\theta > \sqrt{\det(\Gamma)}\kappa(2,2)^{-4}$.

A natural question is to find the critical exponents θ_R and θ_Q, such that

$$\sup_{n\ge 1}\mathbb{E}\exp\Big\{-\theta\frac{(\log n)^2}{n}(R_n - \mathbb{E}R_n)\Big\}\begin{cases} <\infty & \theta < \theta_R \\ =\infty & \theta > \theta_R, \end{cases} \tag{6.3.23}$$

$$\sup_{n\ge 1}\mathbb{E}\exp\Big\{\theta\frac{1}{n}(Q_n - \mathbb{E}Q_n)\Big\}\begin{cases} <\infty & \theta < \theta_Q \\ =\infty & \theta > \theta_Q. \end{cases} \tag{6.3.24}$$

We may ask, in view of the weak laws stated in Theorem 5.4.2, Theorem 5.4.3, and of the critical exponent given by Theorem 4.3.1, if $\theta_R = (2\pi)^{-2}\det(\Gamma)^{-1/2}\kappa(2,2)^{-4}$ and $\theta_Q = \sqrt{\det(\Gamma)}\kappa(2,2)^{-4}$. The following example gives a negative answer to our conjecture on θ_Q.

Example 6.3.4 . *Let N be an arbitrarily large integer. Let the random walk $\{S(n)\}$ is the partial sum of the i.i.d sequence of random vectors in $\mathbb{Z}^d$ that take the values $(N,0),(-N,0),(0,N),(0,-N)$ with probability $(2N^2)^{-1}$ for each, and take the value $(0,0)$ with probability $1-(2/N^2)$. Then Γ is identity matrix. The event that $S(k)=0$ for all $k\le n$ has probability $\big(1-(2/N^2)\big)^n$. On this event $Q_n = n(n-1)/2$. Hence,*

$$\mathbb{E}\exp\Big\{\theta\frac{1}{n}\big(Q_n-\mathbb{E}Q_n\big)\Big\} \ge \exp\Big\{-\theta\frac{\mathbb{E}Q_n}{n}\Big\}\Big(1-\frac{2}{N^2}\Big)^n\exp\Big\{\theta\frac{n-1}{2}\Big\}.$$

By (5.4.1) $\mathbb{E}Q_n = O(n\log n)$. Thus,

$$\theta_Q \le 2\log\Big(1-\frac{2}{N^2}\Big)^{-1}.$$

Since N can be arbitrarily large, so critical exponent θ_Q can be smaller that any previously given positive number.

This example shows that the critical exponent in the continuous setting does not necessarily pass to the discrete setting through weak convergence.

The identification of θ_Q and θ_R remains open.

6.4 Integrability of Q_n and R_n in high dimensions

In an obvious way, the exponential integrabilities for $Q_n-\mathbb{E}Q_n$ and $R_n-\mathbb{E}R_n$ given in Theorem 6.3.2 and Theorem 6.3.3 are closely related to the weak laws stated in Theorem 5.4.2 and Theorem 5.4.3 in the case $d=2$. In this section, we consider the high dimensional setting $d\ge 3$ and control exponential moments for the normalized sequence appearing in Theorem 5.5.3.

Theorem 6.4.1 *As $d=3$,*

$$\sup_n \mathbb{E}\exp\Big\{\frac{\theta}{\sqrt[3]{n\log n}}|Q_n-\mathbb{E}Q_n|^{2/3}\Big\}<\infty \quad \textit{for every } \theta>0. \tag{6.4.1}$$

As $d \ge 4$,

$$\sup_n \mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n}}|Q_n - \mathbb{E}Q_n|^{2/3}\right\} < \infty \quad \textit{for some } \theta > 0. \tag{6.4.2}$$

Proof. We first consider the case $d = 3$. The argument used in the proof of Theorem 6.3.2, applicable only when the normalizing sequence is of the growth rate greater than n^b for some $b > 1/2$, is broken in our setting where the normalizing sequence (of $Q_n - \mathbb{E}Q_n$) is $\sqrt{n \log n}$.

We first prove that for any integer $m \ge 1$,

$$\mathbb{E}|Q_n - \mathbb{E}Q_n|^m = O\Big((n\log n)^{m/2}\Big) \quad (n \to \infty). \tag{6.4.3}$$

We carry out induction on m. By (5.5.5), (6.4.3) holds as $m = 1, 2$. We let $m \ge 3$, assume that it is true for all $1 \le j \le m-1$ and prove it is true for m.

Given n, write $n_1 = [n/2]$, $n_2 = n - n_1$,

$$Q'_{n_2} = \sum_{n_1+1 \le j < k \le n} 1_{\{S_j = S_k\}} \quad \text{and} \quad K_n = \sum_{j=1}^{n_1}\sum_{k=n_1+1}^{n} 1_{\{S_j = S_k\}}.$$

We have that

$$K_n \overset{d}{=} \sum_{j=0}^{n_1-1}\sum_{k=1}^{n-n_1} 1_{\{S_j = \widetilde{S}_k\}}$$

where $\{\widetilde{S}(k)\}$ is an independent copy of $\{S(k)\}$. Taking $d = 3$ and $p = 2$ in (6.2.1) one can see that there is $C_0 > 0$ such that

$$\mathbb{E}|K_n - \mathbb{E}K_n|^m \le C_0^m (m!)^{3/2} n^{m/2} \qquad m, n = 1, 2, \cdots . \tag{6.4.4}$$

By independence between Q_{n_1} and Q'_{n_2},

$$\begin{aligned}
&\Big(\mathbb{E}|Q_n - \mathbb{E}Q_n|^m\Big)^{1/m} \qquad (6.4.5)\\
&\le \Big\{\mathbb{E}\Big(|Q_{n_1} - \mathbb{E}Q_{n_1}| + |Q'_{n_2} - \mathbb{E}Q'_{n_2}|\Big)^m\Big\}^{1/m}\\
&+ \Big\{\mathbb{E}|K_n - \mathbb{E}K_n|^m\Big\}^{1/m}\\
&\le \left\{\sum_{j=0}^{m}\binom{m}{j}\mathbb{E}|Q_{n_1} - \mathbb{E}Q_{n_1}|^j \cdot \mathbb{E}|Q'_{n_2} - \mathbb{E}Q'_{n_2}|^{m-j}\right\}^{1/m}\\
&+ C_0(m!)^{3/(2m)}\sqrt{n}.
\end{aligned}$$

Combining this with the induction assumption,

$$\begin{aligned}&\Big(\mathbb{E}|Q_n-\mathbb{E}Q_n|^m\Big)^{1/m}\\&\le\Big\{O\Big((n\log n)^{m/2}\Big)+\mathbb{E}|Q_{n_1}-\mathbb{E}Q_{n_1}|^m+\mathbb{E}|Q'_{n_2}-\mathbb{E}Q'_{n_2}|^m\Big\}^{1/m}\\&+O(\sqrt{n}).\end{aligned}$$

Write, for each $k\ge 1$,

$$\alpha_k=\sup\Big\{(2^k\log 2^k)^{-1/2}\Big(\mathbb{E}|Q_n-\mathbb{E}Q_n|^m\Big)^{1/m};\quad 2^k\le n\le 2^{k+1}\Big\}.$$

Then,

$$\alpha_{k+1}\le\Big\{O(1)+2^{-\frac{m-2}{2}}\alpha_k^m\Big\}^{1/m}+o(1)\le 2^{-\frac{m-2}{2m}}\alpha_k+O(1)\qquad(k\to\infty).$$

By the fact that $m\ge 3$ one can see that the sequence α_k is bounded. We have proved (6.4.3).

We now claim that there is $C>0$ such that

$$\mathbb{E}|Q_n-\mathbb{E}Q_n|^m\le C^m(m!)^{3/2}(n\log n)^{m/2}\quad m,n=1,2,\cdots. \tag{6.4.6}$$

Indeed, take m_0 sufficiently large so that

$$\Big(1-2^{-\frac{m-2}{2m}}\Big)^{-1}\frac{(m-1)^{1/m}}{\sqrt{2}}\le\frac{1}{2},\qquad 1-2^{-\frac{m-2}{2m}}\ge\frac{1}{4}$$

for all $m\ge m_0$. By (6.4.3), there is a constant $C>0$ such that for all $j=1,\cdots,m_0$,

$$\mathbb{E}|Q_n-\mathbb{E}Q_n|^j\le C^j(j!)^{3/2}(n\log n)^{j/2}\quad n=1,2,\cdots.$$

We may assume that $C\ge 8C_0$ (Recall that C_0 is given in (6.4.4)). By induction (on m), all we have to prove is that for any $m\ge m_0$, if

$$\mathbb{E}|Q_n-\mathbb{E}Q_n|^j\le C^j(j!)^{3/2}(n\log n)^{j/2}\quad n=1,2,\cdots \tag{6.4.7}$$

for every $j=1,\cdots,m-1$ then

$$\mathbb{E}|Q_n-\mathbb{E}Q_n|^m\le C^m(m!)^{3/2}(n\log n)^{m/2}\quad n=1,2,\cdots. \tag{6.4.8}$$

From (6.4.5) and (6.4.7) we have

$$\begin{aligned}&\Big(\mathbb{E}|Q_n-\mathbb{E}Q_n|^m\Big)^{1/m}\\&\le\Big\{2^{-m/2}C^m(n\log n)^{m/2}\sum_{j=1}^{m-1}\binom{m}{j}(j!)^{3/2}((m-j)!)^{3/2}\\&+\mathbb{E}|Q_{n_1}-\mathbb{E}Q_{n_1}|^m+\mathbb{E}|Q'_{n_2}-\mathbb{E}Q'_{n_2}|^m\Big\}^{1/m}+C_0\sqrt{n}(m!)^{3/(2m)}.\end{aligned}$$

Notice that

$$\sum_{j=1}^{m-1}\binom{m}{j}(j!)^{3/2}((m-j)!)^{3/2}=m!\sum_{j=1}^{m-1}(j!)^{1/2}((m-j)!)^{1/2}\le(m!)^{3/2}(m-1)$$

and that by (6.4.3)

$$\beta_m\equiv\sup_{n\ge 1}\Big\{(n\log n)^{-m/2}\mathbb{E}|Q_n-\mathbb{E}Q_n|^m\Big\}<\infty.$$

for every $m\ge 1$. We have

$$\begin{aligned}\beta_m^{1/m}&\le\Big\{\frac{m-1}{2^{m/2}}(m!)^{3/2}C^m+2^{-\frac{m-2}{2}}\beta_m\Big\}^{1/m}+C_0(m!)^{3/(2m)}\\&\le C\frac{(m-1)^{1/m}}{\sqrt{2}}(m!)^{3/(2m)}+2^{-\frac{m-2}{2m}}\beta_m^{1/m}+C_0(m!)^{3/(2m)}.\end{aligned}$$

Hence,

$$\begin{aligned}\beta_m^{1/m}&\le\Big(1-2^{-\frac{m-2}{2m}}\Big)^{-1}\Big\{\frac{(m-1)^{1/m}}{\sqrt{2}}C+C_0\Big\}(m!)^{3/(2m)}\\&\le\Big(\frac{1}{2}C+4C_0\Big)(m!)^{3/(2m)}\le C(m!)^{3/(2m)}.\end{aligned}$$

Therefore, (6.4.3) holds.

By (6.4.3) and Taylor expansion there is $\theta_0>0$ such that

$$\sup_n\mathbb{E}\exp\Big\{\frac{\theta_0}{\sqrt[3]{n\log n}}|Q_n-\mathbb{E}Q_n|^{2/3}\Big\}<\infty. \tag{6.4.9}$$

It remains to extend (6.4.9) to any $\theta>0$. Indeed, for any $\theta>\theta_0$, one can find an integer l such that for any n there is an integer partition $0=n_0<n_1,\cdots,<n_l=n$ such that $n_i-n_{i-1}<n(\theta_0/\theta)^3$ $(i=1,\cdots,l)$. Write

$$Q_n=\sum_{i=1}^{l}\sum_{n_{i-1}<j,k\le n_i}1_{\{S_j=S_k\}}+\sum_{i=2}^{l}\sum_{j=1}^{n_{i-1}}\sum_{k=n_{i-1}+1}^{n_i}1_{\{S_j=S_k\}}. \tag{6.4.10}$$

Observe that for each $2\le i\le l$,

$$\sum_{j=1}^{n_{i-1}}\sum_{k=n_{i-1}+1}^{n_i}1_{\{S_j=S_k\}}\overset{d}{=}\sum_{j=1}^{n_{i-1}}\sum_{k=1}^{n_i-n_{i-1}}1_{\{S_j=S'_k\}}\le I_n.$$

By (6.2.1) (with $d=3$ and $p=2$) and Taylor expansion there is $\lambda>0$ such that

$$\sup_{n\ge1}\mathbb{E}\exp\Big\{\frac{\lambda}{\sqrt[3]{n}}\Big(\sum_{j=1}^{n_{i-1}}\sum_{k=n_{i-1}+1}^{n_i}1_{\{S_j=S_k\}}\Big)^{2/3}\Big\}<\infty.$$

By triangular inequality and Hölder inequality, therefore,

$$\sup_{n\geq 1}\mathbb{E}\exp\left\{\frac{\lambda}{\sqrt[3]{n\log n}}\left(\sum_{i=2}^{l}\sum_{j=1}^{n_{i-1}}\sum_{k=n_{i-1}+1}^{n_i}1_{\{S_j=S_k\}}\right)^{2/3}\right\}<\infty$$

for every $\lambda>0$.

By independence and by triangular inequality

$$\begin{aligned}&\mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n\log n}}\left(\sum_{i=1}^{l}\left|\sum_{n_{i-1}<j,k\leq n_i}1_{\{S_j=S_k\}}-\mathbb{E}\sum_{n_{i-1}<j,k\leq n_i}1_{\{S_j=S_k\}}\right|\right)^{2/3}\right\}\\ &\leq\prod_{i=1}^{l}\mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n\log n}}\left|Q_{n_i-n_{i-1}}-\mathbb{E}Q_{n_i-n_{i-1}}\right|^{2/3}\right\}\\ &\leq\prod_{i=1}^{l}\mathbb{E}\exp\left\{\frac{\theta_0}{\sqrt[3]{(n_i-n_{i-1})\log(n_i-n_{i-1})}}\left|Q_{n_i-n_{i-1}}-\mathbb{E}Q_{n_i-n_{i-1}}\right|^{2/3}\right\}.\end{aligned}$$

Hence, the desired (6.4.1) follows from (6.4.9) and (6.4.10).

The proof of (6.4.2) can be carried out in a similar way, for the fact that the inequality

$$\mathbb{E}\left[\sum_{j,k=1}^{n}1_{\{S(j)=\widetilde{S}(k)\}}\right]^m\leq C^m(m!)^{3/2}n^{m/2}\qquad m,n=1,\cdots \tag{6.4.11}$$

can be extended to the case $d\geq 4$ (Consequently, we have (6.4.4)). Indeed, similar to (6.2.9),

$$\mathbb{E}\left[\sum_{j,k=1}^{n}1_{\{S(j)=\widetilde{S}(k)\}}\right]^m\leq C^m(m!)^2\left[\sum_{j,k=1}^{n}\mathbb{P}\{S(j)=\widetilde{S}(k)\}\right]^m.$$

By (5.1.6) one can show that as $n\to\infty$,

$$\sum_{j,k=1}^{n}\mathbb{P}\{S(j)=\widetilde{S}(k)\}=\sum_{j,k=1}^{n}\mathbb{P}\{S(k+j)=0\}=\begin{cases}O(\log n) & d=4\\ O(1) & d\geq 5.\end{cases}$$

In particular, if we replace the right hand side by $O(\sqrt{n})$ we obtain the estimate

$$\mathbb{E}\left[\sum_{j,k=1}^{n}1_{\{S(j)=\widetilde{S}(k)\}}\right]^m\leq C^m(m!)^2n^{m/2}$$

By the argument used in the proof of Theorem 6.2.1, the above bound can be strengthened into (6.4.11). □

By slightly modifying our argument, Theorem 6.4.1 can be extended to the renormalized range $R_n - \mathbb{E}R_n$. We give the following theorem without proof.

Theorem 6.4.2 *As $d = 3$,*

$$\sup_n \mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n\log n}}|R_n - \mathbb{E}R_n|^{2/3}\right\} < \infty \quad \textit{for every } \theta > 0. \tag{6.4.12}$$

As $d \geq 4$,

$$\sup_n \mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n}}|R_n - \mathbb{E}R_n|^{2/3}\right\} < \infty \quad \textit{for some } \theta > 0. \tag{6.4.13}$$

We now address integrabilities of $\max_{1\leq k\leq n}|Q_k - \mathbb{E}Q_k|$ and $\max_{1\leq k\leq n}|R_k - \mathbb{E}R_k|$. On the one hand, we are not able to provide some sharp maximal inequality in this context so the the integrabilities $\max_{1\leq k\leq n}|Q_k - \mathbb{E}Q_k|$ and $\max_{1\leq k\leq n}|R_k - \mathbb{E}R_k|$ immediately follow from those of $|Q_n - \mathbb{E}Q_n|$ and $|R_n - \mathbb{E}R_n|$. On the other hand, by using entropy method in Section D of Appendix, we are able to prove that $\max_{1\leq k\leq n}|Q_k - \mathbb{E}Q_k|$ and $\max_{1\leq k\leq n}|R_k - \mathbb{E}R_k|$ have essentially same integrabilities as their marginals.

Theorem 6.4.3 *As $d = 3$,*

$$\sup_n \mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n\log n}}\max_{1\leq k\leq n}|Q_n - \mathbb{E}Q_n|^{2/3}\right\} < \infty \quad \forall\theta > 0, \tag{6.4.14}$$

$$\sup_n \mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n\log n}}\max_{1\leq k\leq n}|R_n - \mathbb{E}R_n|^{2/3}\right\} < \infty \quad \forall\theta > 0. \tag{6.4.15}$$

As $d \geq 4$,

$$\sup_n \mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n}}\max_{1\leq k\leq n}|Q_n - \mathbb{E}Q_n|^{2/3}\right\} < \infty \quad \textit{for some } \theta > 0, \tag{6.4.16}$$

$$\sup_n \mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n}}\max_{1\leq k\leq n}|R_n - \mathbb{E}R_n|^{2/3}\right\} < \infty \quad \textit{for some } \theta > 0. \tag{6.4.17}$$

Proof. For similarity we only prove (6.4.14). We first claim that there is a $C > 0$ such that for any n_1, n_2, any $m \geq 1$,

$$\begin{aligned}&\mathbb{E}|(Q_{n_2} - \mathbb{E}Q_{n_2}) - (Q_{n_1} - \mathbb{E}Q_{n_1})|^m \\ &\leq (m!)^{3/2} C^m \max\{n_1, n_2\}^{m/4} (\log \max\{n_1, n_2\})^{m/2} |n_2 - n_1|^{m/4}.\end{aligned} \tag{6.4.18}$$

Assume $n_1 < n_2$.

$$Q_{n_2} - Q_{n_1} = \sum_{n_1 < j < k \leq n_2} 1_{\{S(j)=S(k)\}} + \sum_{j=1}^{n_1} \sum_{k=n_1+1}^{n_2} 1_{\{S(j)=S(k)\}}$$

and observe that

$$\begin{aligned}\sum_{j=1}^{n_1} \sum_{k=n_1+1}^{n_2} 1_{\{S(j)=S(k)\}} &\overset{d}{=} \sum_{j=0}^{n_1-1} \sum_{k=1}^{n_2-n_1} 1_{\{S(j)=\widetilde{S}(k)\}} \\ &= \sum_{x \in \mathbb{Z}^3} l_0(n_1 - 1, x) \widetilde{l}(n_2 - n_1, x)\end{aligned}$$

where $\{\widetilde{S}(k)\}$ is an independent copy of $\{S(k)\}$ and

$$l_0(n, x) = \sum_{k=0}^{n} 1_{\{S(k)=x\}}, \qquad \widetilde{l}(n, x) = \sum_{k=1}^{n} 1_{\{\widetilde{S}(k)=x\}}.$$

By Cauchy-Schwartz inequality,

$$\begin{aligned}&\mathbb{E}\bigg[\sum_{x \in \mathbb{Z}^3} l_0(n_1 - 1, x) \widetilde{l}(n_2 - n_1, x)\bigg]^m \\ &= \sum_{x_1, \cdots, x_m \in \mathbb{Z}^3} \bigg[\mathbb{E} \prod_{k=1}^{m} l_0(n_1 - 1, x_k)\bigg] \bigg[\mathbb{E} \prod_{k=1}^{m} l(n_2 - n_1, x_k)\bigg] \\ &\leq \bigg\{ \sum_{x_1, \cdots, x_m \in \mathbb{Z}^3} \bigg[\mathbb{E} \prod_{k=1}^{m} l_0(n_1 - 1, x_k)\bigg]^2 \bigg\}^{1/2} \\ &\qquad \times \bigg\{ \sum_{x_1, \cdots, x_m \in \mathbb{Z}^3} \bigg[\mathbb{E} \prod_{k=1}^{m} l(n_2 - n_1, x_k)\bigg]^2 \bigg\}^{1/2} \\ &= \Big\{\mathbb{E}\bar{I}_{n_1-1}^m\Big\}^{1/2} \Big\{\mathbb{E}I_{n_2-n_1}^m\Big\}^{1/2}\end{aligned}$$

where

$$\bar{I}_n = \sum_{j=0}^{n} \sum_{k=1}^{n} 1_{\{S(j)=\widetilde{S}(k)\}}, \qquad I_n = \sum_{j=1}^{n} \sum_{k=1}^{n} 1_{\{S(j)=\widetilde{S}(k)\}}.$$

By the moment bound given in (6.2.1) of Theorem 6.2.1 (with $d = 3$ and $p = 2$), therefore, there is a constant $c_1 > 0$ independent of n_1, n_2, such that for any $m \geq 1$,

$$\mathbb{E}\bigg[\sum_{x\in\mathbb{Z}^3} l_0(n_1-1,x)\widetilde{l}(n_2-n_1,x)\bigg]^m \leq (m!)^{3/2}c_1^m n_1^{m/4}(n_2-n_1)^{m/4}.$$

Notice that

$$\sum_{n_1<j<k\leq n_2} 1_{\{S(j)=S(k)\}} \stackrel{d}{=} Q_{n_2-n_1}.$$

By Theorem 6.4.1, there is a constant $c_2 > 0$ independent of n_1, n_2 and $m \geq 1$ such that

$$\begin{aligned}&\mathbb{E}\bigg|\sum_{n_1<j<k\leq n_2} 1_{\{S(j)=S(k)\}} - \mathbb{E}\sum_{n_1<j<k\leq n_2} 1_{\{S(j)=S(k)\}}\bigg|^m\\ &\leq (m!)^{3/2}c_2^m(n_2-n_1)^{m/2}(\log(n_2-n_1))^{m/2}.\end{aligned}$$

Summarizing our argument, we have proved (6.4.18).

Define the random variable $Z_n(\cdot)$ in $C[0,1]$ by

$$\mathcal{Z}_n(t) = \frac{Q_k - \mathbb{E}Q_k}{\sqrt{n\log n}} + (nt-k)\frac{(Q_{k+1}-\mathbb{E}Q_{k+1}) - (Q_k - \mathbb{E}Q_k)}{\sqrt{n\log n}}$$

if $t \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$ for some $0 \leq k \leq n-1$. We claim that there is a constant $C > 0$ such that

$$\mathbb{E}|\mathcal{Z}_n(s) - \mathcal{Z}_n(t)|^m \leq C^m(m!)^{3/2}|s-t|^{m/4} \tag{6.4.19}$$

for every $m, n \geq 1$ and for any $s, t \in [0,1]$.

Assume that $|s-t| \geq 1/n$. By (6.4.18)

$$\mathbb{E}|\mathcal{Z}_n(s) - \mathcal{Z}_n(t)|^m \leq C^m(m!)^{3/2}n^{-m/4}\big|[ns]-[nt]\big|^{m/4}.$$

Hence, (6.4.19) follows from the fact that as $|s-t| \geq 1/n$,

$$\big|[ns]-[nt]\big| \leq 3n|s-t|.$$

We now consider the case $|s-t| < 1/n$. If s, t is in the same interval $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, then

$$\mathcal{Z}_n(s) - \mathcal{Z}_n(t) = n(s-t)\frac{(Q_{k+1}-\mathbb{E}Q_{k+1}) - (Q_k - \mathbb{E}Q_k)}{\sqrt{n\log n}}.$$

By (6.4.18),

$$\begin{aligned}
&\mathbb{E}|\mathcal{Z}_n(s)-\mathcal{Z}_n(t)|^m\\
&=|s-t|^m\Big(\frac{n}{\log n}\Big)^{m/2}\mathbb{E}\Big|(Q_{k+1}-\mathbb{E}Q_{k+1})-(Q_k-\mathbb{E}Q_k)\Big|^m\\
&\le |s-t|^m\Big(\frac{n}{\log n}\Big)^{m/2}(m!)^{3/2}C^m n^{m/4}(\log n)^{m/2}\\
&\le C^m(m!)^{3/2}|s-t|^{m/4}.
\end{aligned}$$

Assume that $|s-t|<1/n$ and s,t are in two consecutive intervals. We may only consider the case $s<t$. Let $s\in\Big[\frac{k-1}{n},\frac{k}{n}\Big]$ and $t\in\Big[\frac{k}{n},\frac{k+1}{n}\Big]$. We have

$$\begin{aligned}
\mathcal{Z}_n(t)-\mathcal{Z}_n(s)&=(nt-k)\frac{(Q_{k+1}-\mathbb{E}Q_{k+1})-(Q_k-\mathbb{E}Q_k)}{\sqrt{n\log n}}\\
&\qquad+(k-ns)\frac{(Q_k-\mathbb{E}Q_k)-(Q_{k-1}-\mathbb{E}Q_{k-1})}{\sqrt{n\log n}}.
\end{aligned}$$

Observe that $0\le(nt-k),(k-ns)\le n(t-s)$. Applying (6.4.18) to each of the two parts on the right hand side gives (6.4.19).

By (6.4.19) and Taylor expansion,

$$\sup_{n\ge1}\sup_{\substack{s,t\in[0,1]\\ s\neq t}}\mathbb{E}\exp\Big\{c\frac{|\mathcal{Z}(s)-\mathcal{Z}(t)|^{2/3}}{|s-t|^{1/6}}\Big\}<\infty.$$

By Theorem D.6 in Appendix, there is a $\theta>0$ such that

$$\sup_{n\ge1}\mathbb{E}\exp\Big\{\theta\sup_{s,t\in[0,1]}|\mathcal{Z}(s)-\mathcal{Z}(t)|^{2/3}\Big\}<\infty.$$

Taking $s=0$ we conclude that (6.4.14) holds for some $\theta>0$. By the same argument used near the end of the proof for (6.4.1), we can extend (6.4.14) to all $\theta>0$. $\square$

6.5 Notes and comments

Section 6.1.

The continuous and discrete settings are quite different when it comes to exponential integrability. The multi-nomial inequalities in Theorem 6.1.1

was established for I_n in [24] and later extended to J_n in [25]. Despite of similarity in multi-nomial form, the argument for Theorem 6.1.2 is quite different. The idea first appeared in Bass, Chen and Rosen [7].

Section 6.2.

The idea of using the moment inequalities to establish exponential integrabilities (Theorem 6.2.1) for I_n and J_n was first appear in Bass, Chen and Rosen [6]. Prior to [6], the exponential integrability of intersection local times was investigated by Brydges and Slade [17] in a different approach.

Section 6.3.

The part "$d = 2$" in Theorem 6.3.2 appears as a discrete version of exponential integrability described in (4.3.3) and was obtained in two papers by Bass, Chen and Rosen ([6], [7]). Prior to that, Brydges and Slade ([17]) had established (6.3.5) in the setting of random walks with continuous time.

Section 6.4.

The main results of this section was established in [30].

Exercise 1. Prove an early result established the following result by Brydges and Slade ([17]): For $d \geq 3$, there is a $\theta_d > 0$ such that

$$\sup_{n\geq 1} \mathbb{E}\exp\left\{\frac{\theta}{n}Q_n\right\} < \infty \qquad \forall \theta < \theta_d.$$

Hint: This is same as

$$\sup_{n\geq 1} \mathbb{E}\exp\left\{\frac{\theta}{n}(Q_n - \mathbb{E}Q_n)\right\} < \infty \qquad \forall \theta < \theta_d.$$

Chapter 7

Large deviations: independent random walks

Our goal in this chapter is to establish the large deviations for the mutual intersection local times run by p independent random walks in $\mathbb{Z}^d$, and for the intersection of the ranges of these random walks. We consider three very different situations: sub-critical dimensions ($p(d-2) < d$), critical-dimensions ($p(d-2) = d$) and super-critical dimensions ($p(d-2) > d$).

The large deviations for I_n and J_n we shall establish in the case $p(d-2) < d$ are also called moderate deviations, since they are essentially discrete version of the large deviation principle for $\alpha\big([0,1]^p\big)$ (Theorem 3.3.2).

In the case $p(d-2) = d$, I_n and J_n are attracted by Γ-distributions (Theorem 5.5.1). Therefore, the moderate deviations for I_n and J_n should yield Γ-tails. On the other hand, the Gagliardo-Nirenberg inequality still holds (and is called Sobolev inequality) in the critical dimensions. This raises the possibility that in addition to the moderate deviations of Γ-tails, I_n and J_n might obey a large deviation principle associated with the Sobolev constant $\kappa(d,p)$.

The independent trajectories intersect at most finitely many times when $p(d-2) > d$. Consequently, the random variables I_∞ and J_∞ are well defined for measuring the intersections in a whole life time. The problem in the super-critical dimensions is to study the large deviations for I_∞ and J_∞.

7.1 Feynman-Kac minorations

Despite of their close relation to the large deviation stated in Theorem 3.3.2, the method of high moment minoration developed in Chapter 3 does not apply directly to the moderate deviations for I_n and J_n for lack of the scaling property in the discrete setting. In our treatment for the moderate deviations given in the next section, the upper bounds rely on the moment inequalities established in (6.1.7) and in (6.1.8) and the laws of weak convergence stated in Theorem 5.3.2 and in Theorem 5.3.4. For the lower bounds, we need to extend the Feynman-Kac large deviation given in Theorem 4.1.6 to the discrete setting. That is the major goal of this section [1].

As usual, $\{S(n)\}$ is a symmetric, square integrable random walk in $\mathbb{Z}^d$ with the smallest supporting group being $\mathbb{Z}^d$.

Theorem 7.1.1 *For any bounded, continuous function f on $\mathbb{R}^d$,*

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\frac{b_n}{n}\sum_{k=1}^n f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\right\}$$
$$\geq\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d} f(\Gamma^{1/2}x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$

where $\mathcal{F}_d$ is the set of the functions g on $\mathbb{R}^d$ with

$$\int_{\mathbb{R}^d}|g(x)|^2dx=1\quad\textit{and}\quad\int_{\mathbb{R}^d}|\nabla g(x)|^2dx<\infty.$$

Proof. We first prove Lemma 7.2 under the aperiodicity assumption on the random walks. Let γ_n be a sequence of odd integers such that $\gamma_n\sim b_n$ $(n\to\infty)$ and write $t_n=[n/\gamma_n]$ Then $t_n\gamma_n\le n\le(t_n+1)\gamma_n$. By the boundedness of f,

$$\sum_{k=1}^n f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)=\sum_{j=1}^{\gamma_n-1}\sum_{k=jt_n+1}^{(j+1)t_n-1}f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)+O(\gamma_n+t_n).$$

Hence, we need only to show that

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\frac{b_n}{n}\sum_{j=1}^{\gamma_n-1}\sum_{k=jt_n+1}^{(j+1)t_n-1}f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\right\}\tag{7.1.1}$$
$$\geq\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d} f(\Gamma^{1/2}x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.$$

[1] Only the lower bounds are installed here. On the other hand, it is very likely that correspondent upper bounds hold.

For each n, define the linear operator $\mathfrak{A}_n$ on $\mathcal{L}^2(\mathbb{Z}^d)$ as

$$\mathfrak{A}_n\xi(x) = \mathbb{E}\bigg(\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}\big(x+S(k)\big)\Big)\bigg\}\xi\big(x+S(t_n)\big)\bigg)$$

where $x \in \mathbb{Z}^d$ and $\xi \in \mathcal{L}^2(\mathbb{Z}^d)$. We claim that $\mathfrak{A}_n$ is symmetric:

$$\langle \xi, \mathfrak{A}_n\eta\rangle = \langle \mathfrak{A}_n\xi, \eta\rangle \qquad \xi, \eta \in \mathcal{L}^2(\mathbb{Z}^d). \tag{7.1.2}$$

Indeed,

$$\begin{aligned}
&\langle \xi, \mathfrak{A}_n\eta\rangle\\
&= \sum_{x\in\mathbb{Z}^d} \xi(x)\mathbb{E}\bigg(\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}\big(x+S(k)\big)\Big)\bigg\}\eta\big(x+S(t_n)\big)\bigg)\\
&= \mathbb{E}\bigg(\sum_{x\in\mathbb{Z}^d} \xi(x)\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}\big(x+S(k)\big)\Big)\bigg\}\eta\big(x+S(t_n)\big)\bigg)\\
&= \mathbb{E}\bigg(\sum_{x\in\mathbb{Z}^d} \xi\big(x-S(t_n)\big)\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}\big(x+S(k)-S(t_n)\big)\Big)\bigg\}\eta(x)\bigg)\\
&= \sum_{x\in\mathbb{Z}^d} \eta(x)\mathbb{E}\bigg(\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}\big(x+S(k)-S(t_n)\big)\Big)\xi\big(x-S(t_n)\big)\bigg)
\end{aligned}$$

where the third equality follows from translation invariance.

Observe that

$$\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}\big(x+S(k)-S(t_n)\big)\Big) = \sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}\big(x+S(t_n-k)-S(t_n)\big)\Big).$$

If we write

$$\bar{S}(k) = S(t_n-k) - S(t_n) \qquad k = 1, \cdots, t_n - 1$$

by symmetry of the random walk we have

$$\{\bar{S}(k)\}_{1\le k\le t_n-1} \stackrel{d}{=} \{S(k)\}_{1\le k\le t_n-1}.$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\bigg(\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}\big(x+S(k)-S(t_n)\big)\Big)\bigg\}\xi\big(x-S(t_n)\big)\bigg)\\
&= \mathbb{E}\bigg(\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}\big(x+S(k)\big)\Big)\bigg\}\xi\big(x+S(t_n)\big)\bigg).
\end{aligned}$$

Summarizing what we have proved,

$$\langle \xi, \mathfrak{A}_n \eta\rangle = \sum_{x\in\mathbb{Z}^d} \eta(x)\mathbb{E}\bigg(\exp\Big\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}(x+S(k))\Big)\Big\}\xi\big(x+S(t_n)\big)\bigg)$$

which is (7.1.2).

By the boundedness of f, it is easy to see that for each n, $\mathfrak{A}_n$ is bounded. Consequently, $\mathfrak{A}_n$ is self-adjoint.

Let $g\in\mathcal{S}(\mathbb{R}^d)$ be supported by a finite box $[-M, M]^d$ and assume that

$$\int_{\mathbb{R}^d} |g(x)|^2 dx = 1.$$

Write

$$\xi_n(x) = \bigg(\sum_{y\in\mathbb{Z}^d} g^2\Big(\sqrt{\frac{b_n}{n}}y\Big)\bigg)^{-1/2} g\Big(\sqrt{\frac{b_n}{n}}x\Big) \qquad x\in\mathbb{Z}^d.$$

By increment independence,

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\frac{b_n}{n}\sum_{j=1}^{\gamma_n-1}\sum_{k=jt_n+1}^{(j+1)t_n-1} f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\Big\}\\
&=\sum_{x\in\mathbb{Z}^d}\mathbb{P}\{S(t_n)=x\}\mathbb{E}\exp\Big\{\frac{b_n}{n}\sum_{j=1}^{\gamma_n-1}\sum_{k=(j-1)t_n+1}^{jt_n-1} f\Big(\sqrt{\frac{b_n}{n}}(x+S(k))\Big)\Big\}\\
&\geq \frac{1+o(1)}{||g||_\infty}\bigg(\sum_{y\in\mathbb{Z}^d} g^2\Big(\sqrt{\frac{b_n}{n}}y\Big)\bigg)\sum_{x\in\mathbb{Z}^d}\mathbb{P}\{S(t_n)=x\}\xi_n(x)\\
&\times\mathbb{E}\bigg(\exp\Big\{\frac{b_n}{n}\sum_{j=1}^{\gamma_n-1}\sum_{k=(j-1)t_n+1}^{jt_n-1} f\Big(\sqrt{\frac{b_n}{n}}(x+S(k))\Big)\Big\}\\
&\qquad\qquad\times\xi_n\big(x+S((\gamma_n-1)t_n)\big)\bigg)\\
&=\frac{1+o(1)}{||g||_\infty}\bigg(\sum_{y\in\mathbb{Z}^d} g^2\Big(\sqrt{\frac{b_n}{n}}y\Big)\bigg)\sum_{x\in\mathbb{Z}^d}\mathbb{P}\{S(t_n)=x\}\xi_n(x)\mathfrak{A}_n^{\gamma_n-1}(\xi_n)(x).
\end{aligned}$$

Notice that

$$\lim_{n\to\infty}\Big(\frac{b_n}{n}\Big)^{d/2}\sum_{y\in\mathbb{Z}^d} g^2\Big(\sqrt{\frac{b_n}{n}}y\Big) = \int_{\mathbb{R}^d} g^2(x)dx = 1.$$

By (5.1.2) (with n being replaced by t_n) in Theorem 5.1.1[2]

$$\mathbb{P}\{S(t_n)=x\}\geq c\Big(\frac{b_n}{n}\Big)^{d/2}\qquad x\in\Big[-M\sqrt{\frac{n}{b_n}},\ M\sqrt{\frac{n}{b_n}}\Big]^d.$$

Since $\xi_n(x)=0$ outside the box

$$\Big[-M\sqrt{\frac{n}{b_n}},\ M\sqrt{\frac{n}{b_n}}\Big]^d$$

there is a $\delta>0$ such that

$$\begin{aligned}&\mathbb{E}\exp\Big\{\frac{b_n}{n}\sum_{j=1}^{\gamma_n-1}\sum_{k=jt_n+1}^{(j+1)t_n-1}f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\Big\}\\&\geq\delta\sum_{x\in\mathbb{Z}^d}\xi_n(x)T_n^{\gamma_n-1}(\xi_n)(x)=\delta\langle\xi_n,\mathfrak{A}_n^{\gamma_n-1}\xi_n\rangle.\end{aligned}$$

By Theorem E.1 in Appendix, the self-adjoint operator $\mathfrak{A}_n$ admits the spectral integral representation

$$\mathfrak{A}_n=\int_{-\infty}^{\infty}\lambda E(d\lambda).$$

By Corollary E.4 in Appendix,

$$\mathfrak{A}_n^{\gamma_n-1}=\int_{-\infty}^{\infty}\lambda^{\gamma_n-1}E(d\lambda).$$

By (E.15) in Appendix, the above representations lead to

$$\langle\xi_n,\mathfrak{A}_n\xi_n\rangle=\int_{-\infty}^{\infty}\lambda\mu_{\xi_n}(d\lambda),$$

$$\langle\xi_n,\mathfrak{A}_n^{\gamma_n-1},\xi_n\rangle=\int_{-\infty}^{\infty}\lambda^{\gamma_n-1}.\mu_{\xi_n}(d\lambda)$$

Notice ((E.5), Appendix) that $\mu_{\xi_n}(\mathbb{R})=1$. That means that the spectral measure μ_{ξ_n} is a probability measure on $\mathbb{R}$. By, Jensen inequality

$$\langle\xi_n,\mathfrak{A}_n^{\gamma_n-1}\xi_n\rangle\geq\Big(\int_{-\infty}^{\infty}\lambda\mu_{\xi_n}(d\lambda)\Big)^{\gamma_n-1}=\langle\xi_n,\mathfrak{A}_n\xi_n\rangle^{\gamma_n-1}$$

Thus,

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\frac{b_n}{n}\sum_{k=t_n+1}^{\gamma_n t_n}f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\Big\}\geq\liminf_{n\to\infty}\log\langle\xi_n,\mathfrak{A}_n\xi_n\rangle.$$

[2] This is where we use the extra assumption on aperiodicity

Notice that

$$\begin{aligned}
&\langle \xi_n, \mathfrak{A}_n \xi_n\rangle \\
&= \sum_{x\in\mathbb{Z}^d} \xi_n(x)\mathbb{E}\bigg(\exp\Big\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(\sqrt{\frac{b_n}{n}}(x+S(k))\Big)\Big\}\xi_n(x+S(t_n))\bigg) \\
&= \big(1+o(1)\big)\int_{\mathbb{R}^d} dx g(x) \\
&\times \mathbb{E}\bigg(\exp\Big\{\frac{b_n}{n}\sum_{k=1}^{t_n-1} f\Big(x+\sqrt{\frac{b_n}{n}}S(k)\Big)\Big\} g\Big(x+\sqrt{\frac{b_n}{n}}S(t_n)\Big)\bigg) \\
&\longrightarrow \int_{\mathbb{R}^d} g(x)\mathbb{E}\bigg(\exp\Big\{\int_0^1 f\big(x+\Gamma^{1/2}W(t)\big)dt\Big\} g\big(x+\Gamma^{1/2}W(1)\big)\bigg)dx
\end{aligned}$$

as $n\to\infty$, where the last step partially follows from invariant principle.

By the variable substitution $x\mapsto \Gamma^{1/2}x$, the integral on the right hand side is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^d} \widetilde{g}(x)\mathbb{E}\bigg(\exp\Big\{\int_0^1 \bar{f}\big(x+W(t)\big)dt\Big\}\widetilde{g}\big(x+W(1)\big)\bigg)dx \\
&\ge \exp\Big\{\int_{\mathbb{R}^d} \bar{f}(x)\widetilde{g}^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla \widetilde{g}(x)|^2dx\Big\}
\end{aligned}$$

where

$$\bar{f}(x) = f\big(\Gamma^{1/2}x\big), \qquad \widetilde{g}(x) = \det(\Gamma)^{1/4} g(\Gamma^{1/2}x)$$

and where the inequality follows from (4.1.25) (with $t=1$; and with f and g being replaced by $\bar{f}$ and $\widetilde{g}$, respectively.).

Summarizing our argument,

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\frac{b_n}{n}\sum_{j=1}^{\gamma_n-1}\sum_{k=jt_n+1}^{(j+1)t_n-1} f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\Big\} \\
&\ge \int_{\mathbb{R}^d} f(\Gamma^{1/2}x)\widetilde{g}^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla\widetilde{g}(x)|^2dx.
\end{aligned}$$

Notice that the set of $\widetilde{g}$ is dense in $\mathcal{F}_d$. Taking supremum over $\widetilde{g}$ on the right hand side leads to the desired conclusion (7.1.1) and therefore to Theorem 7.1.1.

We now drop the aperiodicity condition by performing resolvent approximation. Recall that $0<\theta<1$ be fixed and let $\{\delta_n\}$ is a Bernoulli sequence independent of $\{S(n)\}$ such that

$$\mathbb{P}\{\delta_n = 1\} = 1-\mathbb{P}\{\delta_n=0\} = \theta.$$

Write $\tau_0 = 0$ and

$$\tau_n = \min\{k > \tau_{n-1};\quad \delta_k = 1\} \qquad n = 1, 2, \cdots .$$

Then $\{\tau_n - \tau_{n-1}\}_{n\geq 1}$ is an i.i.d. sequence with common (geometric) distribution

$$\mathbb{P}\{\tau_1 = k\} = \theta(1-\theta)^{k-1} \qquad k = 1, 2, \cdots .$$

Notice that $\{S(\tau_n)\}$ is a random walk whose increment has the distribution

$$\mathbb{P}\{S(\tau_1) = x\} = \sum_{k=1}^{\infty} \theta(1-\theta)^{k-1}\mathbb{P}\{S(k) = x\}.$$

In particular,

$$\mathrm{Cov}\Big(S(\tau_1), S(\tau_1)\Big) = \mathbb{E}\tau_1 \cdot \Gamma = \theta^{-1}\Gamma$$

and, $\mathbb{P}\{S(\tau_1) = 0\} > 0$ implies that $\{S(\tau_n)\}$ is aperiodic. $\{S(\tau_n)\}$ is called resolvent random walk of $\{S(n)\}$ in literature.

Applying what we have proved to the aperiodic random walk $\{S(\tau_n)\}$,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\bigg\{\frac{b_n}{n}\sum_{k=1}^n f\Big(\sqrt{\frac{b_n}{n}}S(\tau_k)\Big)\bigg\} \qquad (7.1.3)\\
&\geq \sup_{g\in\mathcal{F}_d}\bigg\{\int_{\mathbb{R}^d} f(\theta^{-1/2}\Gamma^{1/2}x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2 dx\bigg\}.
\end{aligned}$$

On the other hand, observe that

$$\sum_{k=1}^n f\Big(\sqrt{\frac{b_n}{n}}S(\tau_k)\Big) = \sum_{k=1}^{\tau_n}\delta_k f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big) \leq \sum_{k=1}^{\tau_n} f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big).$$

Let $\lambda > \theta^{-1}$ be fixed for a moment. We have

$$\begin{aligned}
&\mathbb{E}\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^n f\Big(\sqrt{\frac{b_n}{n}}S(\tau_k)\Big)\bigg\}\\
&\leq \mathbb{E}\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^{[\lambda n]} f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\bigg\}\\
&+ \mathbb{E}\Bigg(\exp\bigg\{\frac{b_n}{n}\sum_{k=1}^n f\Big(\sqrt{\frac{b_n}{n}}S(\tau_k)\Big)\bigg\}1_{\{\tau_n\geq\lambda n\}}\Bigg).
\end{aligned}$$

Notice that the second term on the right hand side is bounded by

$$\exp\Big\{||f||_\infty b_n\Big\}\mathbb{P}\{\tau_n \geq \lambda n\}.$$

By Cramér theorem (Theorem 1.1.5) there is $\delta > 0$ such that

$$\mathbb{P}\{\tau_n \geq \lambda n\} \leq e^{-\delta n}$$

for large n. By (7.1.3), therefore,

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\frac{b_n}{n}\sum_{k=1}^{[\lambda n]}f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\right\}\\ &\geq \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}f(\theta^{-1/2}\Gamma^{1/2}x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.\end{aligned}$$

Letting $\theta \to 1$ on the right hand side gives

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\frac{b_n}{n}\sum_{k=1}^{[\lambda n]}f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\right\}\\ &\geq \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}f(\Gamma^{1/2}x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.\end{aligned}$$

Replacing $[\lambda n]$ by n, and f by $\lambda^{-1}f$ gives,

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\frac{b_n}{n}\sum_{k=1}^{n}f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\right\}\\ &\geq \sup_{g\in\mathcal{F}_d}\left\{\lambda^{-1}\int_{\mathbb{R}^d}f(\Gamma^{1/2}x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.\end{aligned}$$

Finally, letting $\lambda \to 1^+$ on the right hand side proves Theorem 7.1.1. □

In our next major theorem, we establish the Feynman-Kac minoration for the sum over the range.

Theorem 7.1.2 *Let f be a bounded, continuous function on $\mathbb{R}^d$.*

(1) When $d = 2$,

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\frac{b_n\log n}{n}\sum_{x\in S[1,n]}f\Big(\sqrt{\frac{b_n}{n}}x\Big)\right\} \qquad (7.1.4)\\ &\geq \sup_{g\in\mathcal{F}_2}\left\{2\pi\sqrt{\det(\Gamma)}\int_{\mathbb{R}^2}f(\Gamma^{1/2}x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla g(x)|^2dx\right\}\end{aligned}$$

for any positive sequence $\{b_n\}$ satisfying

$$b_n \longrightarrow \infty \quad \text{and} \quad b_n = o(\log n) \qquad (n\to\infty).$$

(2) When $d = 3$,

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{ \frac{b_n}{n} \sum_{x\in S[1,n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big)\right\} \tag{7.1.5}$$
$$\geq \sup_{g\in\mathcal{F}_3} \left\{ \gamma_S \int_{\mathbb{R}^3} f(\Gamma^{1/2}x) g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla g(x)|^2 dx\right\}$$

for any positive sequence $\{b_n\}$ satisfying

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o\Big(n^{1/3}\Big) \qquad (n\to\infty).$$

Compared to Theorem 7.1.1, the proof of Theorem 7.1.2 is much harder: The occupation time we work with in Theorem 7.1.1 satisfies the additivity

$$\sum_{i=1}^{j+k} f\Big(\sqrt{\frac{b_n}{n}}S(i)\Big) = \sum_{i=1}^{j} f\Big(\sqrt{\frac{b_n}{n}}S(i)\Big) + \sum_{i=j+1}^{j+k} f\Big(\sqrt{\frac{b_n}{n}}S(i)\Big) \qquad j,k\geq 1.$$

On the other hand, this is no longer the case for the sum over the range. Instead, we have only the sub-additivity in the sense

$$\sum_{x\in S[1,j+k]} f\Big(\sqrt{\frac{b_n}{n}}x\Big) \leq \sum_{x\in S[1,j]} f\Big(\sqrt{\frac{b_n}{n}}x\Big) + \sum_{x\in S[j+1,j+k]} f\Big(\sqrt{\frac{b_n}{n}}x\Big) \qquad j,k\geq 1$$

which is against our interest as we try to work on the lower bound.

The proof of Theorem 7.1.2 consists of two steps. As the first step we work on the quantity

$$\sum_{j=1}^{\gamma_n-1} \sum_{x\in S(jt_n,(j+1)t_n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big)$$

instead on

$$\sum_{x\in S[1,n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big).$$

where γ_n is a sequence of odd integers such that $\gamma_n \sim b_n$ $(n\to\infty)$ and $t_n = [n/\gamma_n]$.

Lemma 7.1.3 *(1) When $d = 2$,*

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{ \frac{b_n \log n}{n} \sum_{j=0}^{\gamma_n-1} \sum_{x\in S(jt_n,(j+1)t_n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big)\right\} \tag{7.1.6}$$
$$\geq \sup_{g\in\mathcal{F}_2} \left\{ 2\pi\sqrt{\det(\Gamma)} \int_{\mathbb{R}^2} f(\Gamma^{1/2}x) g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^2} |\nabla g(x)|^2 dx\right\}$$

if $\log b_n = o(\log n)$.

(2) When $d \geq 3$,

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\frac{b_n}{n}\sum_{j=0}^{\gamma_n - 1} \sum_{x\in S(jt_n,(j+1)t_n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big)\right\} \tag{7.1.7}$$
$$\geq \sup_{g\in\mathcal{F}_3}\left\{\gamma_S \int_{\mathbb{R}^3} f(\Gamma^{1/2}x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla g(x)|^2 dx\right\}$$

if $b_n = o(n)$.

Proof. Due to similarity we only consider the case $d = 2$. The approach is very close to the one used in the proof of Theorem 7.1.1. By resolvent approximation performed in the proof of Theorem 7.1.1, we may assume the aperiodicity of the random walks. Observe that

$$\Big|\sum_{j=0}^{\gamma_n-1} \sum_{x\in S(jt_n,(j+1)t_n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big) - \sum_{j=0}^{\gamma_n-1} \sum_{x\in S(jt_n,(j+1)t_n)}\Big|$$
$$\leq ||f||_\infty \#\Big\{S(t_n), S(2t_n), \cdots, S(\gamma_n t_n)\Big\} \leq ||f||_\infty \gamma_n.$$

So the right hand side is negligible here. Thus, we need only to establish

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\frac{b_n \log n}{n}\sum_{j=1}^{\gamma_n - 1} \sum_{x\in S(jt_n,(j+1)t_n)} f\Big(\sqrt{\frac{b_n}{n}}x\Big)\right\} \tag{7.1.8}$$
$$\geq \sup_{g\in\mathcal{F}_2}\left\{2\pi\sqrt{\det(\Gamma)} \int_{\mathbb{R}^2} f(\Gamma^{1/2}x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^2} |\nabla g(x)|^2 dx.\right\}.$$

Define the continuous linear operator $\mathfrak{B}_n$ on $\mathcal{L}^2(\mathbb{Z}^d)$ as

$$\mathfrak{B}_n\xi(x) = \mathbb{E}\left(\exp\left\{\frac{b_n\log n}{n}\sum_{y-x\in S[1,t_n-1]} f\Big(\sqrt{\frac{b_n}{n}}y\Big)\right\}\xi\big(x+S(t_n)\big)\right).$$

Similar to (7.1.2), $\mathfrak{B}_n$ is self-adjoint. Let g be a bounded function on $\mathbb{R}^d$ and assume that g is infinitely differentiable, supported by a finite square $[-M, M]^2$ with

$$\int_{\mathbb{R}^2} |g(x)|^2 dx = 1$$

and write

$$\xi_n(x) = \Big(\sum_{y\in\mathbb{Z}^2} g^2\Big(\sqrt{\frac{b_n}{n}}y\Big)\Big)^{-1/2} g\Big(\sqrt{\frac{b_n}{n}}x\Big) \qquad x\in\mathbb{Z}^2.$$

An estimate used in the proof of Theorem 7.1.1 leads to

$$
\begin{aligned}
&\mathbb{E}\exp\Big\{\theta\frac{b_n\log n}{n}\sum_{j=1}^{\gamma_n-1}\sum_{x\in S(jt_n,(j+1)t_n)}f\Big(\sqrt{\frac{b_n}{n}}x\Big)\Big\}\\
&\ge \delta\sum_{x\in\mathbb{Z}^2}\xi_n(x)\mathbb{E}\bigg(\exp\Big\{\theta\frac{b_n\log n}{n}\sum_{j=1}^{\gamma_n-1}\sum_{y-x\in S((j-1)t_n,jt_n)}f\Big(\sqrt{\frac{b_n}{n}}y\Big)\Big\}\\
&\qquad\qquad\times\xi_n\Big(x+S\big((\gamma_n-1)t_n\big)\Big)\bigg)\\
&=\delta\langle\xi_n,\mathfrak{B}_n^{\gamma_n-1}\xi_n\rangle\ge\delta\langle\xi_n,\mathfrak{B}_n\xi_n\rangle^{\gamma_n-1}
\end{aligned}
$$

where the last step follows from the argument based on spectral representation and Jensen inequality.

In addition,

$$
\begin{aligned}
&\langle\xi_n,\mathfrak{B}_n\xi_n\rangle\\
&=\big(1+o(1)\big)\int_{\mathbb{R}^2}dxg(x)\\
&\times\mathbb{E}\bigg(\exp\Big\{\frac{b_n\log n}{n}\sum_{x\in S[1,t_n-1]}f\Big(x+\sqrt{\frac{b_n}{n}}y\Big)\Big\}g\Big(x+\sqrt{\frac{b_n}{n}}S(t_n)\Big)\bigg)\\
&\longrightarrow\int_{\mathbb{R}^2}g(x)\mathbb{E}\bigg(\exp\Big\{(2\pi)\sqrt{\det(\Gamma)}\int_0^1f\big(x+\Gamma^{1/2}W(t)\big)dt\Big\}\\
&\qquad\qquad\times g\big(x+\Gamma^{1/2}W(1)\big)\bigg)dx\qquad(n\to\infty)
\end{aligned}
$$

where the last step follows (5.3.41) in Theorem 5.3.6 with n being replacing by t_n-1 [3].

By (4.1.25) and by integration variable substitution, therefore,

$$
\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\frac{b_n\log n}{n}\sum_{j=1}^{\gamma_n-1}\sum_{x\in S(jt_n,(j+1)t_n)}f\Big(\sqrt{\frac{b_n}{n}}x\Big)\Big\}\\
&\ge 2\pi\sqrt{\det(\Gamma)}\int_{\mathbb{R}^2}f(\Gamma^{1/2}x)\tilde{g}^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\tilde{g}(x)|^2dx
\end{aligned}
$$

where $\tilde{g}(x)=\det(\Gamma)^{1/4}g(\Gamma^{1/2}x)$. Taking supremum over g on the right hand side leads to (7.1.8). □

[3] Our assumption on b_n implies that $\log n\sim\log(t_n-1)$

Observe that

$$\sum_{j=0}^{\gamma_n-1}\sum_{x\in S(jt_n,(j+1)t_n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big)-\sum_{x\in S[1,\gamma_n t_n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big)$$
$$=\sum_{x\in\mathbb{Z}^d} f\Big(\sqrt{\frac{b_n}{n}}x\Big)\bigg\{\sum_{j=0}^{\gamma_n-1} 1_{\{x\in S(jt_n,(j+1)t_n]\}}-1_{\{x\in S[1,\gamma_n t_n]\}}\bigg\}.$$

By the fact that

$$\sum_{j=0}^{\gamma_n-1} 1_{\{x\in S(jt_n,(j+1)t_n]\}}-1_{\{x\in S[1,\gamma_n t_n]\}}\ge 0 \qquad x\in\mathbb{Z}^d$$

$$\bigg|\sum_{j=0}^{\gamma_n-1}\sum_{x\in S(jt_n,(j+1)t_n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big)-\sum_{x\in S[1,\gamma_n t_n]} f\Big(\sqrt{\frac{b_n}{n}}x\Big)\bigg|$$
$$\le ||f||_\infty\bigg\{\sum_{j=0}^{\gamma_n-1}\#\big\{S(jt_n,(j+1)t_n]-\#\big\{S[1,\gamma_n t_n]\big\}\bigg\}$$

Therefore, as the second (and the last step) of the proof of Theorem 7.1.2, we establish the following lemma.

Lemma 7.1.4 *Let b_n be a positive sequence with $b_n\to\infty$ and $b_n=o(n)$. For each n, let $1=n_0<n_1<\cdots<n_{\gamma_n}=n$ be an integer partition of $[1,n]$ such that $\gamma_n\sim b_n$ and $n_j-n_{j-1}\sim n/b_n$ uniformly over $1\le j\le\gamma_n$.*

(1). When $d=2$,

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\bigg\{\theta\frac{b_n\log n}{n}\Big(\sum_{j=1}^{\gamma_n}\#\big\{S(n_{j-1},n_j]\big\}-\#\big\{S[1,n]\big\}\Big)\bigg\}=0$$

for any $\theta>0$, if $b_n=o(\log n)$ $(n\to\infty)$.

(2). When $d=3$,

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\bigg\{\theta\frac{b_n}{n}\Big(\sum_{j=1}^{\gamma_n}\#\big\{S(n_{j-1},n_j]\big\}-\#\big\{S[1,n]\big\}\Big)\bigg\}=0$$

for any $\theta>0$, if $b_n=o(n^{1/3})$ $(n\to\infty)$.

Proof. Due to similarity, we only deal with the case $d=2$. We start with the decomposition

$$\begin{aligned}
&\sum_{j=1}^{\gamma_n}\#\big\{S(n_{j-1},n_j]\big\}-\#\big\{S[1,n]\big\}\\
&=\sum_{j=1}^{\gamma_n}\Big(\#\big\{S(n_{j-1},n_j]\big\}-\mathbb{E}\#\big\{S(n_{j-1},n_j]\big\}\Big)\\
&+\Big(\mathbb{E}\#\big\{S[1,n]\big\}-\#\big\{S[1,n]\big\}\Big)\\
&+\sum_{j=1}^{\gamma_n}\Big(\mathbb{E}\#\big\{S(n_{j-1},n_j]\big\}-\mathbb{E}\#\big\{S[1,n]\big\}\Big).
\end{aligned}$$

By (5.3.39)

$$\sum_{j=1}^{\gamma_n}\Big(\mathbb{E}\#\big\{S(n_{j-1},n_j]\big\}-\mathbb{E}\#\big\{S[1,n]\big\}\Big)=o\Big(\frac{n}{\log n}\Big)\qquad(n\to\infty).$$

By (6.3.4) in Theorem 6.3.2 there is a $c>0$ such that

$$\sup_n\mathbb{E}\exp\left\{c\frac{(\log n)^2}{n}\Big(\mathbb{E}\#\big\{S[1,n]\big\}-\#\big\{S[1,n]\big\}\Big)\right\}<\infty.$$

It remains to show

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\frac{b_n\log n}{n}\left[\sum_{j=1}^{\gamma_n}\Big(\#\big\{S(n_{j-1},n_j]\big\}-\mathbb{E}\#\big\{S(n_{j-1},n_j]\big\}\Big)\right]\right\}=0.$$

This follows from independence

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\frac{b_n\log n}{n}\left[\sum_{j=1}^{\gamma_n}\Big(\#\big\{S(n_{j-1},n_j]\big\}-\mathbb{E}\#\big\{S(n_{j-1},n_j]\big\}\Big)\right]\right\}\\
&=\prod_{j=1}^{\gamma_n}\mathbb{E}\exp\left\{\theta\frac{b_n\log n}{n}\Big(\#\big\{S[1,n_j-n_{j-1}]\big\}-\mathbb{E}\#\big\{S[1,n_j-n_{j-1}]\big\}\Big)\right\}
\end{aligned}$$

and from the fact (implied by (6.3.18) in Theorem 6.3.3) that

$$\lim_{n\to\infty}\mathbb{E}\exp\left\{\theta\frac{b_n\log n}{n}\Big(\#\big\{S[1,n_j-n_{j-1}]\big\}-\mathbb{E}\#\big\{S[1,n_j-n_{j-1}]\big\}\Big)\right\}=1$$

uniformly over $1\le j\le\gamma_n$. □

7.2 Moderate deviations in sub-critical dimensions

Recall that

$$I_n = \#\Big\{(k_1, \cdots, k_p);\quad S_1(k_1) = \cdots = S_p(k_p)\Big\}$$

$$J_n = \#\big\{S_1[1,n] \cap \cdots \cap S_p[1,n]\big\}.$$

Historically, the phrase "moderate deviation" is often used for the large deviation principle associated with some weak law of convergence. Recall, for example, that (Theorem 5.3.2) under $p(d-2) < d$,

$$n^{-\frac{2p-d(p-1)}{2}} I_n \xrightarrow{d} \det(\Gamma)^{-\frac{p-1}{2}} \alpha\big([0,1]^p\big)$$

where

$$\alpha\big([0,t]^p\big) = \int_{\mathbb{R}^d} \bigg[\prod_{j=1}^p \int_0^t \delta_x\big(W_j(s)\big) ds\bigg] dx$$

is the mutual intersection local time run by independent d-dimensional Brownian motions $W_1(t), \cdots, W_p(t)$.

Consequently, for each large but fixed $b > 0$,

$$\mathbb{P}\Big\{I_n \geq \lambda n^{\frac{2p-d(p-1)}{2}} b^{\frac{d(p-1)}{2}}\Big\} \xrightarrow{n\to\infty} \mathbb{P}\Big\{\alpha\big([0,1]^p\big) \geq \lambda \det(\Gamma)^{\frac{p-1}{2}} b^{\frac{d(p-1)}{2}}\Big\}$$

or,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{b} \log \mathbb{P}\Big\{I_n \geq \lambda n^{\frac{2p-d(p-1)}{2}} b^{\frac{d(p-1)}{2}}\Big\} \\ &= \frac{1}{b} \log \mathbb{P}\Big\{\alpha\big([0,1]^p\big) \geq \lambda \det(\Gamma)^{\frac{p-1}{2}} b^{\frac{d(p-1)}{2}}\Big\}. \end{aligned} \tag{7.2.1}$$

By Theorem 3.3.2 (or its version in (3.3.4)),

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{\alpha\big([0,1]^p\big) \geq t^{\frac{d(p-1)}{2}}\Big\} = -\frac{p}{2}\kappa(d,p)^{-\frac{dp}{d(p-1)}}. \tag{7.2.2}$$

Here we recall that $\kappa(d,p) > 0$ is the best constant of Gagliardo-Nirenberg inequality

$$||f||_{2p} \leq C||\nabla f||_2^{\frac{d(p-1)}{2p}} ||f||_2^{1-\frac{d(p-1)}{2p}} \qquad f \in W^{1,2}(\mathbb{R}^d)$$

and

$$W^{1,2}(\mathbb{R}^d) = \big\{f \in \mathcal{L}^2(\mathbb{R}^d);\quad \nabla f \in \mathcal{L}^2(\mathbb{R}^d)\big\}.$$

Let $b \to \infty$ in (7.2.1). By (7.2.2), the right hand side of (7.2.1) converges to

$$-\frac{p}{2}\det(\Gamma)^{1/d}\kappa(d,p)^{\frac{4p}{d(p-1)}}\lambda^{\frac{2}{d(p-1)}}.$$

Therefore, it is not hard to believe (probably not hard to even prove) that

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{I_n \ge \lambda n^{\frac{2p-d(p-1)}{2}} b_n^{\frac{d(p-1)}{2}}\Big\}$$
$$= -\frac{p}{2}\det(\Gamma)^{1/d}\kappa(d,p)^{-\frac{4p}{d(p-1)}}\lambda^{\frac{2}{d(p-1)}}$$

if b_n increases to infinity with sufficiently slow rate.

The general observation we made here does not have any practical value if we can not say anything about the rate of b_n except the words "sufficiently slow". In the direction of moderate deviations, the concern is whether the result includes all "right" sequences b_n or, at least those crucial to applications (such as $b_n = \log\log n$ to the law of the iterated logarithm). Our main results in this section are the moderate deviations for I_n and J_n associated to the weak laws given in Theorem 5.3.2 and in Theorem 5.3.4, respectively.

Theorem 7.2.1 *Under $p(d-2) < d$,*

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{I_n \ge \lambda n^{\frac{2p-d(p-1)}{2}} b_n^{\frac{d(p-1)}{2}}\Big\}$$
$$= -\frac{p}{2}\det(\Gamma)^{1/d}\kappa(d,p)^{-\frac{4p}{d(p-1)}}\lambda^{\frac{2}{d(p-1)}}$$

for any $\lambda > 0$ and for any positive sequence $\{b_n\}$ satisfying

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o(n) \qquad (n \to \infty).$$

Proof. By Theorem 1.2.6, we need only to show that for any $\theta > 0$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m}\Big(\mathbb{E}I_n^m\Big)^{1/p} = \Psi(\theta) \tag{7.2.3}$$

where

$$\Psi(\theta) = \frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}} \tag{7.2.4}$$
$$\times\det(\Gamma)^{-\frac{p-1}{2p-d(p-1)}}\kappa(d,p)^{\frac{4p}{2p-d(p-1)}}\theta^{\frac{2p}{2p-d(p-1)}}.$$

We now work on the upper bound of (7.2.3). Let $t > 0$ be fixed and let $t_n = [tn/b_n]$ and $\gamma_n = [n/t_n]$. Then $n \le t_n(\gamma_n + 1)$. By taking $a = \gamma_n + 1$ in inequality (6.1.8) we have

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m \Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m} \big(\mathbb{E}I_n^m\big)^{1/p}\\
&\le \bigg(\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m \Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m} \big(\mathbb{E}I_{t_n}^m\big)^{1/p}\bigg)^{\gamma_n+1}.
\end{aligned}$$

From Theorem 5.3.2,

$$\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2}} I_{t_n} \xrightarrow{d} t^{\frac{2p-d(p-1)}{2}} \det(\Gamma)^{-\frac{p-1}{2}} \alpha\big([0,1]^p\big)$$

as $n \to \infty$. By (6.2.1) in Theorem 6.2.1 and by dominated convergence,

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m \Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m} \big(\mathbb{E}I_{t_n}^m\big)^{1/p}\\
&\longrightarrow \sum_{m=0}^{\infty} \frac{1}{m!}\theta^m t^{\frac{2p-d(p-1)}{2p}m} \det(\Gamma)^{-\frac{p-1}{2p}m} \Big(\mathbb{E}\alpha\big([0,1]^p\big)^m\Big)^{1/p}.
\end{aligned}$$

Summarizing our argument

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{b_n} \log\bigg(\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m \Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m} \big(\mathbb{E}I_n^m\big)^{1/p}\bigg)\\
&\le \frac{1}{t}\log\bigg(\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m t^{\frac{2p-d(p-1)}{2p}m} \det(\Gamma)^{-\frac{p-1}{2p}m} \Big(\mathbb{E}\alpha\big([0,1]^p\big)^m\Big)^{1/p}\bigg).
\end{aligned}$$

By (7.2.2) and Theorem 1.2.8,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\log\bigg(\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m t^{\frac{2p-d(p-1)}{2p}m} \det(\Gamma)^{-\frac{p-1}{2p}m} \Big(\mathbb{E}\alpha\big([0,1]^p\big)^m\Big)^{1/p}\bigg)\\
&= \sup_{\lambda>0}\Big\{\theta \det(\Gamma)^{-\frac{p-1}{2p}}\lambda - \frac{1}{2}\kappa(d,p)^{\frac{dp}{d(p-1)}}\lambda^{\frac{2}{d(p-1)}}\Big\}\\
&= \Psi(\theta).
\end{aligned}$$

Thus, we have established the upper bound for (7.2.3).

We now prove the lower bound for (7.2.3). Recall the representation:

$$I_n = \sum_{x\in\mathbb{Z}^d} \prod_{j=1}^{p} l_j(n,x).$$

By independence, for any integer $m \geq 1$

$$\mathbb{E}I_n^m = \sum_{x_1,\cdots,x_m \in \mathbb{Z}^d} \left[\mathbb{E}\prod_{k=1}^m l(n,x_k)\right]^p.$$

Let $q > 1$ be the conjugate of p and let f be a bounded, continuous and locally supported function on $\mathbb{R}^d$ such that $||f||_q = 1$. Write

$$C_n = \left(\sum_{x\in\mathbb{Z}^d} \left|f\left(\sqrt{\frac{b_n}{n}}x\right)\right|^q\right)^{1/q}.$$

By Hölder inequality

$$\begin{aligned}\left(\mathbb{E}I_n^m\right)^{1/p} &\geq C_n^{-m} \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d} f\left(\sqrt{\frac{b_n}{n}}x_1\right)\cdots f\left(\sqrt{\frac{b_n}{n}}x_m\right)\mathbb{E}\prod_{k=1}^m l(n,x_k)\\ &= C_n^{-m}\mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} f\left(\sqrt{\frac{b_n}{n}}x\right)l(n,x)\right]^m = C_n^{-m}\mathbb{E}\left[\sum_{k=1}^n f\left(\sqrt{\frac{b_n}{n}}S(k)\right)\right]^m.\end{aligned}$$

Consequently,

$$\begin{aligned}&\sum_{m=0}^\infty \frac{1}{m!}\theta^m\left(\frac{b_n}{n}\right)^{\frac{2p-d(p-1)}{2p}m}\left(\mathbb{E}I_n^m\right)^{1/p}\\ &\geq \mathbb{E}\exp\left\{\theta\left(\frac{b_n}{n}\right)^{\frac{2p-d(p-1)}{2p}} C_n^{-1}\sum_{k=1}^n f\left(\sqrt{\frac{b_n}{n}}S(k)\right)\right\}.\end{aligned}$$

Observe that

$$C_n \sim \left(\frac{n}{b_n}\right)^{\frac{d(p-1)}{2p}}\left(\int_{\mathbb{R}^d}|f(x)|^q dx\right)^{1/q} = \left(\frac{n}{b_n}\right)^{\frac{d(p-1)}{2p}} \qquad (n\to\infty).$$

By Theorem 7.1.1, therefore,

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{b_n}\log\left(\sum_{m=0}^\infty \frac{\theta^m}{m!}\left(\frac{b_n}{n}\right)^{\frac{2p-d(p-1)}{2p}m}\left(\mathbb{E}I_n^m\right)^{1/p}\right)\\ &\geq \sup_{g\in\mathcal{F}_d}\left\{\theta\int_{\mathbb{R}^d} f(\Gamma^{1/2}x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2 dx\right\}.\end{aligned}$$

Notice that the set of bounded, continuous and locally supported functions f is dense on the unit sphere of $\mathcal{L}^q(\mathbb{R}^d)$, and that when f runs over this set, so dose the function

$$f_\Gamma(x) = \det(\Gamma)^{\frac{1}{2q}} f(\Gamma^{1/2}x).$$

Taking supremum over f on the right hand side gives

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{b_n}\log\Big(\sum_{m=0}^\infty\frac{1}{m!}\theta^m\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m}\big(\mathbb{E}I_n^m\big)^{1/p}\Big)\\
&\ge\sup_{g\in\mathcal{F}_d}\Big\{\theta\det(\Gamma)^{-\frac{1}{2q}}\Big(\int_{\mathbb{R}^d}|g(x)|^{2p}dx\Big)^{1/p}-\frac12\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big\}\\
&=\Psi(\theta)
\end{aligned}$$

where the last step follows from Theorem C.1 in Appendix.

We have proved the lower bound for (7.2.3). □

Theorem 7.2.2 *(1). As $d=2$ and $p\ge 2$,*

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{J_n\ge\lambda\frac{n}{(\log n)^p}b_n^{p-1}\Big\}\\
&=-\frac{p}{2}(2\pi)^{-\frac{p}{p-1}}\det(\Gamma)^{-\frac{1}{2(p-1)}}\kappa(2,p)^{-\frac{2p}{p-1}}\lambda^{\frac{1}{p-1}}
\end{aligned}\tag{7.2.5}$$

for any $\lambda>0$ and for any positive sequence b_n satisfying

$$b_n\longrightarrow\infty\quad\text{and}\quad b_n=o(\log n)\qquad(n\to\infty).$$

(2). As $d=3$ and $p=2$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{J_n\ge\lambda\sqrt{nb_n^3}\Big\}=-\det(\Gamma)^{1/3}\gamma^{-4/3}\kappa(3,2)^{-8/3}\lambda^{2/3}\tag{7.2.6}$$

for any $\lambda>0$ and for any positive sequence b_n satisfying

$$b_n\longrightarrow\infty\quad\text{and}\quad b_n=o\big(n^{1/3}\big)\qquad(n\to\infty).$$

Proof. By Theorem 1.2.6, we need only to show that when $d=2$ and $p\ge 2$,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\frac{b_n(\log n)^p}{n}\Big)^{m/p}\Big(\mathbb{E}J_n^m\Big)^{1/p}\\
&=\frac{1}{p}\Big(\frac{2(p-1)}{p}\Big)^{p-1}(2\pi\theta)^p\sqrt{\det(\Gamma)}\kappa(2,p)^{2p}
\end{aligned}\tag{7.2.7}$$

and that when $d=3$ and $p=2$

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{m/4}\Big(\mathbb{E}J_n^m\Big)^{1/2}\\
&=\frac12\Big(\frac34\Big)^3(\gamma_S\theta)^4\det(\Gamma)^{-1}\kappa(3,2)^8.
\end{aligned}\tag{7.2.8}$$

They can be proved in a similar way as the proof of (7.2.3). $\square$

We end this section by the following comment on the restrictions that we put on the positive sequence $\{b_n\}$. By the fact that $I_n \le n^p$, it is unlikely that the tail behavior of the probability

$$\mathbb{P}\{I_n \ge \lambda n^p\}$$

would fall into the pattern described in Theorem 7.2.1. Thus, the restriction "$b_n = o(n)$" is best possible for Theorem 7.2.1.

By the fact that $J_n \le n$, the restriction "$b_n = o\big(n^{1/3}\big)$" is best possible for Theorem 7.2.2 in the case $d = 3$ and $p = 2$.

As for Theorem 7.2.2 in the case $d = 2$ and $p \ge 2$, we prove that for any $\lambda > 2\pi\sqrt{\det(\Gamma)}$ such that that

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\Big\{ J_n \ge \lambda \frac{n}{\log n} \Big\} = -\infty. \tag{7.2.9}$$

This fact shows that Theorem 7.2.2 fails if we take $b_n = \log n$ when $d = 2$.

By the fact that J_n decreases as p increases, we need only to establish (7.2.9) in the case $p = 2$. By the fact that

$$J_n \le \#\big\{S_1[1,n]\big\} = R_n$$

we have

$$\mathbb{P}\Big\{ J_n \ge \lambda \frac{n}{\log n} \Big\} \le \mathbb{P}\Big\{ R_n \ge \lambda \frac{n}{\log n} \Big\} \le \mathbb{P}\Big\{ R_n - \mathbb{E}R_n \ge \epsilon \frac{n}{\log n} \Big\}$$

for any $0 < \epsilon < \lambda - 2\pi\sqrt{\det(\Gamma)}$. Here we use the fact ((5.3.39)) that

$$\mathbb{E}R_n \sim 2\pi\sqrt{\det(\Gamma)} \frac{n}{\log n}.$$

According to (6.3.18) in Theorem 6.3.3, for any $\theta > 0$

$$\sup_n \mathbb{E}\exp\Big\{ \theta \frac{(\log n)^2}{n} (R_n - \mathbb{E}R_n) \Big\} < \infty$$

which leads to (7.2.9) by a standard way of using Chebyshev inequality.

7.3 Laws of the iterated logarithm

In this section we apply the moderate deviations established in Section 7.2 to the following laws of the iterated logarithm.

Theorem 7.3.1 *Under $p(d-2)<d$,*

$$\limsup_{n\to\infty} n^{-\frac{2p-d(p-1)}{2}}(\log\log n)^{-\frac{d(p-1)}{2}} I_n$$
$$=\Big(\frac{2}{p}\Big)^{-\frac{d(p-1)}{2}}\det(\Gamma)^{-\frac{p-1}{2}}\kappa(d,p)^{2p} \quad a.s.$$

Theorem 7.3.2 *(1). As $d=2$ and $p\ge 2$,*

$$\limsup_{n\to\infty}\frac{(\log n)^p}{n(\log\log n)^{p-1}}J_n=(2\pi)^p\Big(\frac{2}{p}\Big)^{p-1}\sqrt{\det(\Gamma)}\kappa(2,p)^{2p} \quad a.s. \tag{7.3.1}$$

(2). As $d=3$ and $p=2$,

$$\limsup_{n\to\infty}\frac{1}{\sqrt{n(\log\log n)^3}}J_n=\gamma_S^2\det(\Gamma)^{-1/2}\kappa(3,2)^4 \quad a.s. \tag{7.3.2}$$

Given the moderate deviations in Section 7.2, the proof of the upper bounds for Theorem 7.3.1 and Theorem 7.3.2 is a standard practice of the Borel-Cantelli lemma. To establish the lower bounds, we introduce the following notation: For any $\bar y=(y_1,\cdots,y_p)\in(\mathbb{Z}^d)^p$, $\mathbb{P}_{\bar y}$ and $\mathbb{E}_{\bar y}$ are, respectively, the distribution and expectation with respect to the trajectories of the random walks $S_1(k),\cdots,S_p(k)$ with the starting points $S_1(0)=y_1,\cdots,S_p(0)=y_p$. In connection with the notation we usually use, $\mathbb{P}\{\cdot\}=\mathbb{P}_{\bar 0}\{\cdot\}$ and $\mathbb{E}\{\cdot\}=\mathbb{E}_{\bar 0}\{\cdot\}$ Set $b_n=\log\log n$ and $t_n=[n/b_n]$.

Examining the proof of Theorem 3.3.7 carefully, all we need is to show that

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\inf_{|\bar y|\le\sqrt{t_n}}\mathbb{P}_{\bar y}\big\{I_n\ge\lambda n^{\frac{2p-d(p-1)}{2}}b_n^{\frac{d(p-1)}{2}}\big\} \tag{7.3.3}$$
$$=-\frac{p}{2}\det(\Gamma)^{1/d}\kappa(d,p)^{-\frac{4p}{d(p-1)}}\lambda^{\frac{2}{d(p-1)}}$$

and that

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\inf_{|\bar y|\le\sqrt{t_n}}\mathbb{P}_{\bar y}\Big\{J_n\ge\lambda\frac{n}{(\log n)^p}b_n^{p-1}\Big\} \tag{7.3.4}$$
$$=-\frac{p}{2}(2\pi)^{-\frac{p}{p-1}}\det(\Gamma)^{-\frac{1}{2(p-1)}}\kappa(2,p)^{-\frac{2p}{p-1}}\lambda^{\frac{1}{p-1}}$$

when $d=2$, $p\ge 2$; and

$$\lim_{n\to\infty}\frac{1}{b_n}\log\inf_{|\bar y|\le\sqrt{t_n}}\mathbb{P}_{\bar y}\Big\{J_n\ge\lambda\sqrt{nb_n^3}\Big\} \tag{7.3.5}$$
$$=-\det(\Gamma)^{1/3}\gamma^{-4/3}\kappa(3,2)^{-8/3}\lambda^{2/3}$$

where $\lambda > 0$ is arbitrary.

Due to similarity we only prove (7.3.3). Notice that for any $\bar{y} = (y_1, \cdots, y_p) \in (\mathbb{Z}^d)^p$,

$$\mathbb{E}_{\bar{y}} I_n^m = \sum_{x_1,\cdots,x_m \in \mathbb{Z}^d} \prod_{j=1}^p \mathbb{E} \prod_{k=1}^m l(n, x_k - y_j). \tag{7.3.6}$$

By Hölder inequality, $\mathbb{E}_{\bar{y}} I_n^m \le \mathbb{E} I_n^m$ for all $\bar{y} \in (\mathbb{Z}^d)^p$. By (7.2.3), therefore,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \sup_{\bar{y}\in(\mathbb{Z}^d)^p} \sum_{m=0}^\infty \frac{\theta^m}{m!} \Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m} \Big(\mathbb{E}_{\bar{y}} I_n^m\Big)^{1/p} = \Psi(\theta)$$

for all $\theta > 0$, where $\Psi(\theta)$ is written in (7.2.4). By a general argument based on Theorem 1.2.6, we will have (7.3.3) if we can show that for each $\theta > 0$,

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \inf_{|\bar{y}|\le\sqrt{t_n}} \sum_{m=0}^\infty \frac{\theta^m}{m!} \Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m} \Big(\mathbb{E}_{\bar{y}} I_n^m\Big)^{1/p} \ge \Psi(\theta). \tag{7.3.7}$$

By the method of resolvent approximation, we may assume that the random walks are aperiodic in our proof of (7.3.7). By (7.3.6),

$$\begin{aligned}\mathbb{E}_{\bar{y}} I_n^m &\ge \sum_{x_1,\cdots,x_m \in \mathbb{Z}^d} \prod_{j=1}^p \mathbb{E} \prod_{k=1}^m \big[l(n, x_k - y_j) - l(t_n, x_k - y_j)\big] \\ &= \sum_{x_1,\cdots,x_m \in \mathbb{Z}^d} \prod_{j=1}^p \sum_{z\in\mathbb{Z}^d} \mathbb{P}\{S(t_n) = z - y_j\} \mathbb{E} \prod_{k=1}^m l(n - t_n, x_k - z)\end{aligned}$$

where the last step follows from increment independence.

Write $B_n = \{x \in \mathbb{Z}^d;\ |x| \le \epsilon\sqrt{t_n}\}$, where $\epsilon > 0$ is fixed but arbitrary. By (5.1.2) in Lemma 5.1.1[4], there is a constant $c > 0$ such that

$$\mathbb{P}\{S(t_n) = x\} \ge c t_n^{-d/2} \qquad \forall x \in 2B_n.$$

[4]This is where the aperiodicity is needed

Consequently,

$$
\begin{aligned}
&\inf_{|\bar{y}|\le\sqrt{t_n}}\mathbb{E}_{\bar{y}}I_n^m\\
&\ge c^pt_n^{-dp/2}\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\prod_{j=1}^p\sum_{z\in B_n}\mathbb{E}\prod_{k=1}^m l(n-t_n,x_k-z)\\
&=c^pt_n^{-dp/2}\sum_{z_1,\cdots,z_p\in B_n}\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\prod_{j=1}^p\mathbb{E}\prod_{k=1}^m l(n-t_n,x_k-z_j)\\
&=c^pt_n^{-dp/2}\sum_{z_1,\cdots,z_p\in B_n}\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p l_j(n-t_n,x-y_j)\Big)^m\\
&\ge c^pt_n^{-dp/2}\#(B_n)^p\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d}\#(B_n)^{-1}\sum_{z_1,\cdots,z_p\in B_n}\prod_{j=1}^p l_j(n-t_n,x-y_j)\Big)^m\\
&=c^pt_n^{-dp/2}\#(B_n)^p\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\zeta_j(n,x)\Big)^m
\end{aligned}
$$

where the fourth inequality follows from Jensen inequality and

$$\zeta_j(n,x)=\#(B_n)^{-1}\sum_{z\in B_n}l_j(n-t_n,x-z).$$

The problem has been reduced from (7.3.7) to

$$
\begin{aligned}
\liminf_{\epsilon\to0^+}\liminf_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}}&\\
\times\bigg\{\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\zeta_j(n,x)\Big)^m\bigg\}^{1/p}&\ge\Psi(\theta).
\end{aligned}
\tag{7.3.8}
$$

Let $f(x)$ be a bounded, uniformly continuous function locally supported on $\mathbb{R}^d$ with $||f||_q=1$. By independence and by similar estimation used in last section

$$
\begin{aligned}
&\bigg\{\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\zeta_j(n,x)\Big)^m\bigg\}^{1/p}=\bigg\{\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{E}\prod_{k=1}^m\zeta(n,x_k)\Big]^p\bigg\}^{1/p}\\
&\ge C_n^{-m}\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}f\Big(\sqrt{\frac{b_n}{n}}x_1\Big)\cdots f\Big(\sqrt{\frac{b_n}{n}}x_m\Big)\mathbb{E}\prod_{k=1}^m\zeta(n,x_k)\\
&=C_n^{-m}\mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}f\Big(\sqrt{\frac{b_n}{n}}x\Big)\zeta(n,x_k)\Big]^m
\end{aligned}
$$

where

$$C_n = \Big\{ \sum_{x\in\mathbb{Z}^d} \Big| f\Big(\sqrt{\frac{b_n}{n}}x\Big)\Big|^q \Big\}^{1/q}.$$

Summing up on the both sides,

$$\begin{aligned}&\sum_{m=0}^\infty \frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m} \Big\{\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p \zeta_j(n,x)\Big)^m\Big\}^{1/p}\\ &\ge \mathbb{E}\exp\Big\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}} C_n^{-1}\sum_{x\in\mathbb{Z}^d} f\Big(\sqrt{\frac{b_n}{n}}x\Big)\zeta(n,x)\Big\}.\end{aligned}$$

On the other hand, set

$$(\mathfrak{D}_\epsilon f)(x) = (K_d\epsilon^d)^{-1}\int_{\{|y|\le\epsilon\}} f(x+y)dy \qquad x\in\mathbb{R}^d$$

where K_d is the volume of a d-dimensional unit ball. By uniform continuity of f,

$$\begin{aligned}&\sum_{x\in\mathbb{Z}^d} f\Big(\sqrt{\frac{b_n}{n}}x\Big)\zeta(n,x)\\ &= \#(B_n)^{-1}\sum_{x\in\mathbb{Z}^d} l(n-t_n,x)\sum_{z\in B_n} f\Big(\sqrt{\frac{b_n}{n}}(x+z)\Big)\\ &= \#(B_n)^{-1}\sum_{k=1}^{n-t_n}\sum_{z\in B_n} f\Big(\sqrt{\frac{b_n}{n}}(S(k)+z)\Big)\\ &= o(n) + \sum_{k=1}^{n-t_n}(\mathfrak{D}_\epsilon f)\Big(\sqrt{\frac{b_n}{n}}S(k)\Big).\end{aligned}$$

By the fact that $C_n \sim (n/b_n)^{d/2q}$, and by Theorem 7.1.1 (with f being replaced by $\mathfrak{D}_\epsilon f$),

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty \frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m} \Big\{\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p \zeta_j(n,x)\Big)^m\Big\}^{1/p}\\ &\ge \sup_{g\in\mathcal{F}_d}\Big\{\theta\int_{\mathbb{R}^d}\mathfrak{D}_\epsilon f(\Gamma^{1/2}x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big\}.\end{aligned}$$

Letting $\epsilon\to 0^+$ on the both sides gives that

$$\begin{aligned}&\liminf_{\epsilon\to0^+}\liminf_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty \frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m} \Big\{\mathbb{E}\Big(\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p \zeta_j(n,x)\Big)^m\Big\}^{1/p}\\ &\ge \sup_{g\in\mathcal{F}_d}\Big\{\theta\int_{\mathbb{R}^d} f(\Gamma^{1/2}x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big\}.\end{aligned}$$

Taking supremum over f on the right hand side leads to (7.3.8). □

7.4 What do we expect in critical dimensions?

The laws of the iterated logarithm for I_n and J_n in the critical dimensions have been obtained in Marcus and Rosen ([128], [144]).

Theorem 7.4.1 *Assume additionally that the i.i.d. increments of the random walks have finite third moment.*

(1). When $d = 4$ and $p = 2$,

$$\limsup_{n\to\infty} \frac{I_n}{n \log\log\log n} = \frac{1}{2\pi^2\sqrt{\det(\Gamma)}} \quad a.s. \tag{7.4.1}$$

$$\limsup_{n\to\infty} \frac{J_n}{n \log\log\log n} = \frac{\gamma_S^2}{2\pi^2\sqrt{\det(\Gamma)}} \quad a.s. \tag{7.4.2}$$

(2). When $d = p = 3$,

$$\limsup_{n\to\infty} \frac{I_n}{n \log\log\log n} = \frac{1}{2\pi^2 \det(\Gamma)} \quad a.s. \tag{7.4.3}$$

$$\limsup_{n\to\infty} \frac{J_n}{n \log\log\log n} = \frac{\gamma_S^3}{2\pi^2 \det(\Gamma)} \quad a.s. \tag{7.4.4}$$

We skip the proof. The reader is referred to Marcus and Rosen [128] for the proof of Part (1) and to Rosen [144] for Part (2).

If we interpret the results given in Theorem 7.4.1 in terms of large deviations, or if we exam the proofs given in Marcus-Rosen [128] and Rosen [144], we would see that in the critical dimensions (defined by $p(d-2) = d$),

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ I_n \ge \lambda b_n \log n \Big\} = -C_1(d,p)\lambda \tag{7.4.5}$$

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ J_n \ge \lambda b_n \log n \Big\} = -C_2(d,p)\lambda \tag{7.4.6}$$

for small scale b_n ($b_n = \log\log\log n$ in the papers [128] and [144]).

In (7.4.5) and (7.4.6), $I_n/\log n$ and $J_n/\log n$ have the tails asymptotically same as the tails of Γ-distributions. This behavior is due to the fact (Theorem 5.5.1) that $I_n/\log n$ and $J_n/\log n$ weakly converges to Γ-distributions. In this regard, (7.4.5) and (7.4.6) are indeed moderate deviations.

On the other hand, it is highly unlikely that (7.4.5) and (7.4.5) hold for any b_n growing faster than $\log n$. What has not been known is the forms of the large deviations, if any, for I_n and J_n when the deviation scale b_n is large. We notice, however, that all moderate deviations for I_n in sub-critical dimensions involve with the best constant $\kappa(d,p)$ of the Gagliardo-Nirenberg inequality

$$||f||_{2p} \le C||\nabla f||_2^{\frac{d(p-1)}{2p}} ||f||_2^{1-\frac{d(p-1)}{2p}} \qquad f \in W^{1,2}(\mathbb{R}^d).$$

Further, notice the case $p(d-2) = d$ is critical to the inequality. Indeed, the Gagliardo-Nirenberg inequality becomes Sobolev inequality

$$||f||_{2p} \le C||\nabla f||_2 \qquad f \in W^{1,2}(\mathbb{R}^d).$$

In view of (7.4.5), (7.4.6) and Theorem 7.2.2, based on the facts that $b_n^p = b_n \log n$ as $b_n = (\log n)^{\frac{1}{p-1}}$ and that $I_n \le n^p$, $J_n \le n$, we make the following conjecture.

Conjecture 7.4.2 *When $p(d-2) = d$, the large deviation principle*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ I_n \ge \lambda b_n^p \Big\} = -\frac{p}{2} \det(\Gamma)^{1/d} \kappa(d,p)^{-2} \lambda^{1/p} \tag{7.4.7}$$

holds for $\lambda > 0$ and for positive sequence b_n with

$$\lim_{n\to\infty} b_n/(\log n)^{\frac{1}{p-1}} = \infty \quad \text{and} \quad b_n = o(n).$$

Similarly,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ J_n \ge \lambda b_n^p \Big\} = -\gamma_S^p \frac{p}{2} \det(\Gamma)^{1/d} \kappa(d,p)^{-2} \lambda^{1/p} \tag{7.4.8}$$

holds for $\lambda > 0$ and for positive sequence b_n with

$$\lim_{n\to\infty} b_n/(\log n)^{\frac{1}{p-1}} = \infty \quad \text{and} \quad b_n = o\big(n^{1/p}\big).$$

7.5 Large deviations in super-critical dimensions

We now move to the super-critical dimensions known as $p(d-2) > d$. In this case,

$$I_\infty \equiv \#\big\{(k_1, \cdots, k_p); \quad S_1(k_1) = \cdots = S_p(k_p)\big\} < \infty \qquad a.s.$$

$$J_\infty = \#\big\{S_1[1,\infty)\cap\cdots\cap S_p[1,\infty)\big\} < \infty \qquad a.s.$$

We are interested in the tail probabilities

$$\mathbb{P}\{I_\infty \ge t\} \quad \text{and} \quad \mathbb{P}\{J_\infty \ge t\}$$

as $t\to\infty$. In this section we shall investigate the tail behavior of I_∞.

Theorem 7.5.1 *There is $-\infty < L < \infty$ such that*

$$\lim_{k\to\infty}\frac{1}{k}\log\frac{1}{(k!)^p}\mathbb{E}I_\infty^k = L. \tag{7.5.1}$$

Consequently,

$$\lim_{t\to\infty} t^{-1/p}\log\mathbb{P}\big\{I_\infty \ge t\big\} = -pe^{-L/p}. \tag{7.5.2}$$

Proof. We need only to show (7.5.1) as (7.5.2) is a consequence of (7.5.1) (Theorem 1.2.7). To this end we define the "augmented" intersection local time $\tilde{I}_\infty$ by

$$\tilde{I}_\infty = \sum_{i_1,\cdots,i_p=0}^{\infty} 1_{\{S_1(i_1)=\cdots=S(i_p)\}}.$$

That is, we include the intersection at time 0. We first claim that there is an extended number $-\infty \le L < \infty$ such that

$$\lim_{k\to\infty}\frac{1}{k}\log\frac{1}{(k!)^p}\mathbb{E}\tilde{I}_\infty^k = L. \tag{7.5.3}$$

By Lemma 1.3.1, it suffices for (7.5.3) to show that

$$\Big(\mathbb{E}\tilde{I}_\infty^{k+l}\Big)^{1/p} \le \binom{k+l}{k}\Big(\mathbb{E}\tilde{I}_\infty^{k}\Big)^{1/p}\Big(\mathbb{E}\tilde{I}_\infty^{l}\Big)^{1/p} \tag{7.5.4}$$

holds for any integers $k,l\ge 1$.

Write

$$l(x) = \sum_{j=0}^{\infty} 1_{\{S(k)=x\}}.$$

Then ((5.5.9) ensures that the quantities we deal with in the following calculation are finite.)

$$\begin{aligned}
\mathbb{E}\tilde{I}_\infty^{k+l} &= \sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d}\left[\mathbb{E}\prod_{j=1}^{k+l} l(x_j)\right]^p \\
&= \sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d}\left\{\sum_{i_1,\cdots,i_{k+l}=0}^{\infty}\mathbb{E}\prod_{j=1}^{k+l}1_{\{S(i_j)=x_j\}}\right\}\left[\mathbb{E}\prod_{j=1}^{k+l} l(x_j)\right]^{p-1} \\
&= \sum_{i_1,\cdots,i_{k+l}=0}^{\infty}\sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d}\left\{\mathbb{E}\prod_{j=1}^{k+l}1_{\{S(i_j)=x_j\}}\right\}\left[\mathbb{E}\prod_{j=1}^{k+l} l(x_j)\right]^{p-1}.
\end{aligned}$$

For any two vectors $(j_1,\ldots,j_k)$ and $(j_{k+1},\ldots,j_{k+l})$ we write

$$(j_1,\ldots,j_k)\prec(j_{k+1},\ldots,j_{k+l})\Longleftrightarrow\max\{j_1,\ldots,j_k\}\le\min\{j_{k+1},\ldots,j_{k+l}\}.$$

For each vector $(i_1,\cdots,i_{k+l})$ we reorder its component so the resulting vector $(i'_1,\ldots,i'_{k+l})$ satisfies

$$(i'_1,\ldots,i'_k)\prec(i'_{k+1},\ldots,i'_{k+l})$$

Let the permutation $\sigma\in\Sigma_{k+l}$ be such that $i'_{\sigma(j)}=i_j$ $(j=1,\cdots,k+l)$. Then

$$\begin{aligned}
&\sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d}\left\{\mathbb{E}\prod_{j=1}^{k+l}1_{\{S(i_j)=x_j\}}\right\}\left[\mathbb{E}\prod_{j=1}^{k+l} l(x_j)\right]^{p-1} \\
&= \sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d}\left\{\mathbb{E}\prod_{j=1}^{k+l}1_{\{S(i'_j)=x_{\sigma(j)}\}}\right\}\left[\mathbb{E}\prod_{j=1}^{k+l} l(x_j)\right]^{p-1} \\
&= \sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d}\left\{\mathbb{E}\prod_{j=1}^{k+l}1_{\{S(i'_j)=x_j\}}\right\}\left[\mathbb{E}\prod_{j=1}^{k+l} l(x_{\sigma^{-1}(j)})\right]^{p-1} \\
&= \sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d}\left\{\mathbb{E}\prod_{j=1}^{k+l}1_{\{S(i'_j)=x_j\}}\right\}\left[\mathbb{E}\prod_{j=1}^{k+l} l(x_j)\right]^{p-1}
\end{aligned}$$

where σ^{-1} is the inverse of σ and the last step follows from the permutation invariance

$$\prod_{j=1}^{k+l} l(x_{\sigma^{-1}(j)})=\prod_{j=1}^{k+l} l(x_j).$$

There are at most $\binom{k+l}{k}$ different ways to partition the (not necessarily distinct) indexes $i_1,\cdots,i_{k+l}$ into two groups $\{i'_1,\cdots,i'_k\}$ and $\{i'_{k+1},\cdots,i'_{k+l}\}$.

Thus,

$$\begin{aligned}
&\mathbb{E}\tilde{I}_\infty^{k+l}\\
&\le \binom{k+l}{k} \sum_{\substack{(i_1,\ldots,i_k)\\ \prec(i_{k+1},\ldots,i_{k+l})}} \sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d} \left[\mathbb{E}\prod_{j=1}^{k+l} 1_{\{S(i_j)=x_j\}}\right]\left[\mathbb{E}\prod_{j=1}^{k+l} l(x_j)\right]^{p-1}\\
&= \binom{k+l}{k} \sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d} \left[\sum_{\substack{(i_1,\ldots,i_k)\\ \prec(i_{k+1},\ldots,i_{k+l})}} \mathbb{E}\prod_{j=1}^{k+l} 1_{\{S(i_j)=x_j\}}\right]\left[\mathbb{E}\prod_{j=1}^{k+l} l(x_j)\right]^{p-1}.
\end{aligned}$$

Applying Hölder inequality on the right hand side

$$\begin{aligned}
&\mathbb{E}\tilde{I}_\infty^{k+l}\\
&\le \binom{k+l}{k}\left\{\sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d} \left[\sum_{\substack{(i_1,\ldots,i_k)\\ \prec(i_{k+1},\ldots,i_{k+l})}} \mathbb{E}\prod_{j=1}^{k+l} 1_{\{S(i_j)=x_j\}}\right]^p\right\}^{1/p}\left(\mathbb{E}\tilde{I}_\infty^{k+l}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Consequently,

$$\left(\mathbb{E}\tilde{I}_\infty^{k+l}\right)^{1/p} \le \binom{k+l}{k}\left\{\sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d} \left[\sum_{\substack{(i_1,\ldots,i_k)\\ \prec(i_{k+1},\ldots,i_{k+l})}} \mathbb{E}\prod_{j=1}^{k+l} 1_{\{S(i_j)=x_j\}}\right]^p\right\}^{1/p}.$$

For each $i_1,\cdots,i_{k+l}$ with $(i_1,\ldots,i_k)\prec(i_{k+1},\ldots,i_{k+l})$, we set $i_*=\max\{i_1,\cdots,i_k\}$. By increment independence,

$$\mathbb{E}\prod_{j=1}^{k+l} 1_{\{S(i_j)=x_j\}} = \left\{\mathbb{E}\prod_{j=1}^{k} 1_{\{S(i_j)=x_j\}}\right\}\left\{\mathbb{E}\prod_{j=k+1}^{k+l} 1_{\{S(i_j-i_*)=x_j-x_{i_*}\}}\right\}.$$

Hence,

$$\begin{aligned}
&\sum_{\substack{(i_1,\ldots,i_k)\\ \prec(i_{k+1},\ldots,i_{k+l})}} \mathbb{E}\prod_{j=1}^{k+l} 1_{\{S(i_j)=x_j\}}\\
&= \sum_{i_1,\cdots,i_k=0}^{\infty}\left\{\mathbb{E}\prod_{j=1}^{k} 1_{\{S(i_j)=x_j\}}\right\} \sum_{i_{k+1},\cdots,i_{k+l}=0}^{\infty} \mathbb{E}\prod_{j=k+1}^{k+l} 1_{\{S(i_j)=x_j-x_{i_*}\}}\\
&= \sum_{i_1,\cdots,i_k=0}^{\infty}\left\{\mathbb{E}\prod_{j=1}^{k} 1_{\{S(i_j)=x_j\}}\right\}\mathbb{E}\prod_{j=k+1}^{k+l} l(x_j-x_{i_*}).
\end{aligned}$$

By triangular inequality, therefore, for any fixed $(x_1, \cdots, x_k)$

$$\Big\{ \sum_{x_{k+1},\cdots,x_{k+l}\in\mathbb{Z}^d} \Big[\sum_{\substack{(i_1,\ldots,i_k) \\ \prec(i_{k+1},\ldots,i_{k+l})}} \mathbb{E}\prod_{j=1}^{k+l} 1_{\{S(i_j)=x_j\}}\Big]^p\Big\}^{1/p}$$
$$\le \sum_{i_1,\cdots,i_k=0}^{\infty}\Big[\mathbb{E}\prod_{j=1}^{k} 1_{\{S(i_j)=x_j\}}\Big]\Big\{\sum_{x_{k+1},\cdots,x_{k+l}\in\mathbb{Z}^d}\Big[\mathbb{E}\prod_{j=k+1}^{k+l} l(x_j-x_{i_*})\Big]^p\Big\}^{1/p}.$$

By translation invariance,

$$\sum_{x_{k+1},\cdots,x_{k+l}\in\mathbb{Z}^d}\Big[\mathbb{E}\prod_{j=k+1}^{k+l} l(x_j-x_{i_*})\Big]^p = \sum_{x_{k+1},\cdots,x_{k+l}\in\mathbb{Z}^d}\Big[\mathbb{E}\prod_{j=k+1}^{k+l} l(x_j)\Big]^p = \mathbb{E}\tilde{I}_\infty^l.$$

Consequently,

$$\sum_{x_{k+1},\cdots,x_{k+l}\in\mathbb{Z}^d} \Big[\sum_{\substack{(i_1,\ldots,i_k) \\ \prec(i_{k+1},\ldots,i_{k+l})}} \mathbb{E}\prod_{j=1}^{k+l} 1_{\{S(i_j)=x_j\}}\Big]^p$$
$$\le \mathbb{E}\tilde{I}_\infty^l\Big[\sum_{i_1,\cdots,i_k=0}^{\infty}\mathbb{E}\prod_{j=1}^{k} 1_{\{S(i_j)=x_j\}}\Big]^p = \mathbb{E}\tilde{I}_\infty^l\Big[\mathbb{E}\prod_{j=1}^{k} l(x_j)\Big]^p.$$

Summing over $(x_1, \cdots, x_k)$ on the both sides,

$$\sum_{x_1,\cdots,x_{k+l}\in\mathbb{Z}^d} \Big[\sum_{\substack{(i_1,\ldots,i_k) \\ \prec(i_{k+1},\ldots,i_{k+l})}} \mathbb{E}\prod_{j=1}^{k+l} 1_{\{S(i_j)=x_j\}}\Big]^p$$
$$\le \Big(\mathbb{E}\tilde{I}_\infty^l\Big)\sum_{x_1,\cdots,x_k\in\mathbb{Z}^d}\Big[\mathbb{E}\prod_{j=1}^{k} l(x_j)\Big]^p = \Big(\mathbb{E}\tilde{I}_\infty^l\Big)\Big(\mathbb{E}\tilde{I}_\infty^k\Big).$$

Summarizing our argument, we have proved (7.5.4).

As the second step, we show that $L > -\infty$. Indeed, for each $m \ge 1$,

$$\mathbb{E}\tilde{I}_\infty^k \ge \mathbb{E}I_\infty^k = \sum_{x_1,\cdots,x_k\in\mathbb{Z}^d}\Big[\mathbb{E}\prod_{j=1}^{k}\sum_{i=1}^{\infty} 1_{\{S(i)=x_j\}}\Big]^p$$
$$\ge \sum_{x_1,\cdots,x_k\in\mathbb{Z}^d}\Big[\sum_{\substack{i_1,\ldots,i_k \\ \text{distinct}}}\mathbb{P}\big\{S(i_1)=x_1,\cdots,S(i_k)=x_k\big\}\Big]^p$$
$$= \sum_{x_1,\cdots,x_k\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_k}\prod_{j=1}^{k}\mathfrak{G}\big(x_{\sigma(j)}-x_{\sigma(j-1)}\big)\Big]^p$$

where $\mathfrak{G}(x)$ is the *Green's function*

$$\mathfrak{G}(x) := \sum_{k=1}^{\infty} \mathbb{P}\{S(k) = x\} \qquad x \in \mathbb{Z}^d.$$

Hence, by Theorem 3.1.1,

$$L \geq p \log \sup \left\{ \sum_{x,y \in \mathbb{Z}^d} \mathfrak{G}(x-y) f(x) f(y); \quad \sum_{x \in \mathbb{Z}^d} |f(x)|^{\frac{2p}{2p-1}} = 1 \right\} \tag{7.5.5}$$
$$> -\infty.$$

Finally, we come to the proof of (7.5.1). Noticing that $I_\infty \leq \tilde{I}_\infty$, by (7.5.3) we only have to prove that

$$\liminf_{k\to\infty} \frac{1}{k} \log \frac{1}{(k!)^p} \mathbb{E} I_\infty^k \geq L.$$

Notice that

$$\tilde{I}_\infty \leq I_\infty + \sum_{j=1}^{p} l_j(0)$$

where

$$l_j(x) = \sum_{k=0}^{\infty} 1_{\{S_j(k)=x\}}.$$

So we have

$$\left(\mathbb{E}\tilde{I}_\infty^k\right)^{1/k} \leq \left(\mathbb{E} I_\infty^k\right)^{1/k} + p\left(\mathbb{E} l^k(0)\right)^{1/k}.$$

For the second term,

$$\begin{aligned}
\mathbb{E} l^k(0) &= \sum_{i_1,\cdot,i_k=0}^{\infty} \mathbb{P}\big\{S(i_1) = \cdots = S(i_k) = 0\big\} \\
&\leq k! \sum_{i_1 \leq \cdots \leq i_k} \mathbb{P}\big\{S(i_1) = \cdots = S(i_k) = 0\big\} \\
&= k! \sum_{i_1 \leq \cdots \leq i_k} \prod_{j=1}^{k} \mathbb{P}\{S(i_j - i_{j-1}) = 0\} \\
&\leq k! \left(\sum_{i=0}^{\infty} \mathbb{P}\{S(i) = 0\} \right)^k.
\end{aligned}$$

This confirms our claim. $\square$

Our next investigation is to identify the constant appearing on the right-hand side of (7.5.2) in terms of certain variation. Unfortunately, the relation posted in (7.5.5) can not be further developed into an equality.

To introduce the variational representation, recall the Green's function

$$\mathfrak{G}(x) := \sum_{k=1}^{\infty} \mathbb{P}\{S(k) = x\} \qquad x \in \mathbb{Z}^d.$$

Notice that we are following the (slightly unusual) convention of not summing over the time $k = 0$, which has an influence on the value of $\mathfrak{G}(0)$. A crucial fact in the super-critical dimensions is (5.5.9) that

$$\sum_{x\in\mathbb{Z}^d} \mathfrak{G}^p(x) < \infty. \tag{7.5.6}$$

Lemma 7.5.2 *For every non-negative $h \in \mathcal{L}^q(\mathbb{Z}^d)$, the linear map*

$$\mathfrak{K}_h g(x) = \sqrt{e^{h(x)} - 1} \sum_{y\in\mathbb{Z}^d} \mathfrak{G}(x-y) g(y) \sqrt{e^{h(y)} - 1} \qquad g \in \mathcal{L}^2(\mathbb{Z}^d)$$

defines a bounded, self-adjoint operator on $\mathcal{L}^2(\mathbb{Z}^d)$.

Further,

$$||\mathfrak{K}_h|| = \sup_{|g|_2=1} \langle g, \mathfrak{K}_h g\rangle \le ||\mathfrak{G}||_p \Big(e^{||h||_q} - 1\Big). \tag{7.5.7}$$

Proof. By the symmetry of the random walk, it is easy to see that for $g_1, g_2 \in \mathcal{L}^2(\mathbb{Z}^d)$

$$\langle g_1, \mathfrak{K}_h g_2\rangle = \langle \mathfrak{K}_h g_1, g_2\rangle$$

if convergence is not a problem.

For any $g \in \mathcal{L}^2(\mathbb{Z}^d)$, write

$$f(x) = g(x)\sqrt{e^{h(x)} - 1} \qquad x \in \mathbb{Z}^d.$$

By Hölder inequality

$$\sum_{x\in\mathbb{Z}^d} |f(x)|^{\frac{2p}{2p-1}} \le \Big\{\sum_{x\in\mathbb{Z}^d} g^2(x)\Big\}^{\frac{p}{2p-1}} \Big\{\sum_{x\in\mathbb{Z}^d} \Big(e^{h(x)} - 1\Big)^q\Big\}^{\frac{p-1}{2p-1}}.$$

By Taylor expansion and by triangular inequality

$$\Big\{\sum_{x\in\mathbb{Z}^d} \Big(e^{h(x)} - 1\Big)^q\Big\}^{1/q} \le \sum_{k=1}^{\infty} \frac{1}{k!} ||h||_q^k = e^{||h||_q} - 1.$$

Thus,

$$\sum_{x\in\mathbb{Z}^d}|f(x)|^{\frac{2p}{2p-1}} \le \Big(e^{||h||_q}-1\Big)^{\frac{p}{2p-1}}||g||_2^{\frac{2p}{2p-1}}.$$

Consequently,

$$\begin{aligned}\big|\langle g,\mathfrak{K}_h g\rangle\big| &= \sum_{x,y\in\mathbb{Z}^d}\mathfrak{G}(x-y)|f(x)||f(y)| \le \Big\{\sum_{x\in\mathbb{Z}^d}\mathfrak{G}^p(x)\Big\}^{1/p}||f||^2_{\frac{2p}{2p-1}}\\ &\le \Big\{\sum_{x\in\mathbb{Z}^d}\mathfrak{G}^p(x)\Big\}^{1/p}\Big(e^{||h||_q}-1\Big)||g||_2^2.\end{aligned}$$

where the second step follows from the estimation used in the proof of Lemma 3.2.1. In particular, the above estimation gives the following bound to the spectral radius

$$||\mathfrak{K}|| = \sup_{||g||_2=1}\big|\langle g,\mathfrak{K}_h g\rangle\big| \le \Big\{\sum_{x\in\mathbb{Z}^d}\mathfrak{G}^p(x)\Big\}^{1/p}\Big(e^{||h||_q}-1\Big).$$

Finally, the equality in (7.5.7) follows from the obvious fact that

$$\sup_{||g||_2=1}\big|\langle g,\mathfrak{K}_h g\rangle\big| = \sup_{||g||_2=1}\langle g,\mathfrak{K}_h g\rangle.$$

□

Our next theorem is formulated in terms of the variation problem

$$\mathcal{C}(S) = \inf\Big\{||h||_q;\quad h\ge 0 \text{ and } ||\mathfrak{K}_h||\ge 1\Big\}. \tag{7.5.8}$$

We claim that

$$\log\Big(1+||\mathfrak{G}||_p^{-1}\Big)\le \mathcal{C}(S)\le \varrho_S^{-1}$$

where

$$\varrho_S = \sup\Big\{\sum_{x,y\in\mathbb{Z}^d}\mathfrak{G}(x-y)f(x)f(y);\quad \sum_{x\in\mathbb{Z}^d}|f(x)|^{\frac{2p}{2p-1}}=1\Big\}.$$

Indeed, (7.5.6) and (7.5.7) imply the lower bound. On the other hand, for any $f(x)\ge 0$ with

$$\sum_{x\in\mathbb{Z}^d}f^{\frac{2p}{2p-1}}(x)=1$$

write $h(x) = \varrho_S^{-1} f^{\frac{2(p-1)}{2p-1}}(x)$ and $g(x) = f^{\frac{p}{2p-1}}(x)$. We have $||h||_q = \varrho_S^{-1}$ and $||g||_2 = 1$. Hence, by the inequality $e^\theta - 1 \geq \theta$ ($\theta \geq 0$),

$$\begin{aligned} ||\mathfrak{K}_h|| \geq \langle g, \mathfrak{K}_h g\rangle &\geq \sum_{x,y} \mathfrak{G}(x-y)\sqrt{h(x)}g(x)\sqrt{h(y)}g(y) \\ &= \varrho_S^{-1} \sum_{x,y} \mathfrak{G}(x-y) f(x) f(y). \end{aligned}$$

Taking supremum over f on the right hand side gives that $||\mathfrak{K}_h|| \geq 1$ which leads to the desired upper bound $\mathcal{C}(S) \leq \varrho_S^{-1}$.

Theorem 7.5.3 *The constant L in Theorem 7.5.1 is equal to $p\log\mathcal{C}(S)$. Consequently,*

$$\lim_{t\to\infty} t^{-1/p} \log \mathbb{P}\big\{I_\infty \geq t\big\} = -p\mathcal{C}(S).$$

We spend the rest of the section to prove this theorem. That is, we shall establish that

$$\lim_{k\to\infty} \frac{1}{k} \log \frac{1}{(k!)^p} \mathbb{E} I_\infty^k = -p\log\mathcal{C}(S). \tag{7.5.9}$$

To do that, we need to represent $\mathbb{E}I_\infty^k$ in a tractable form. What we get turns out to be much more complicated than the Le Gall's identity given in Theorem 2.2.8. Notice that

$$\mathbb{E}I_\infty^k = \sum_{x_1,\cdots,x_k\in\mathbb{Z}^d} \bigg[\mathbb{E} \sum_{i_1,\cdots,i_k=1}^\infty \prod_{l=1}^k 1_{\{S(i_l)=x_l\}}\bigg]^p.$$

We now face the challenge in reordering the tuple $(i_1, \cdots, i_k)$, for some of $i_1, \cdots, i_k$ may take same value. Indeed, we will miss the right constant by over-counting (summing over $i_1 \leq \cdots \leq i_k$) and by under-counting (summing over $i_1 < \cdots < i_k$).

We define, for any $1 \leq m \leq k$, the family of all m-partitions of the set $\{1, \ldots, k\}$ as

$$\begin{aligned} \mathcal{E}_m := \Big\{ \pi = (\pi_1, \ldots, \pi_m);\quad \pi_j \neq \emptyset \ \text{ with } \ \pi_i \cap \pi_j = \emptyset \\ \text{for all } i \neq j \text{ and } \bigcup_{j=1}^m \pi_j = \{1, \ldots, k\}\Big\}, \end{aligned}$$

where we assume that the elements $\pi_1, \ldots, \pi_m$ of an m-partition are ordered by increasing order of their minimal elements. To any $(i_1, \ldots, i_k)$ we associate

- a tuple $(i_1^*, \ldots, i_m^*)$ of distinct natural numbers such that
$$\{i_1, \ldots, i_k\} = \{i_1^*, \ldots, i_m^*\}$$
and elements in the tuple $(i_1^*, \ldots, i_m^*)$ appear in the order in which they appear first in $(i_1, \ldots, i_k)$;
- an m-partition $(\pi_1, \ldots, \pi_m) \in \mathcal{E}_m$ with $i_j = i_\ell^*$ whenever $j \in \pi_\ell$.

Conversely, given a tuple $(j_1, \ldots, j_m)$ of distinct natural numbers and an m-partition π we can find $(i_1, \ldots, i_k)$ such that the induced m-tuple is $(j_1, \ldots, j_m)$ and the induced m-partition is π.

For any $\pi \in \mathcal{E}_m$, define

$$\mathcal{A}(\pi) = \left\{(x_1, \ldots, x_k) \in (\mathbb{Z}^d)^k \,:\, x_i = x_j \text{ for all } i, j \in \pi_\ell \text{ and } \ell \in \{1, \ldots, m\}\right\}.$$

For any $(x_1, \ldots, x_k) \in \mathcal{A}(\pi)$ and for any $1 \le \ell \le m$, we use x_{π_ℓ} for the common value of $\{x_j \,:\, j \in \pi_\ell\}$. There are two facts which we shall use frequently in the later argument:

First, for any $j_1, \cdots, j_m \ge 1$ with $j_1 + \cdots + j_m = k$

$$\#\Big\{\pi = (\pi_1, \cdots, \pi_m) \in \mathcal{E}_m; \quad \#(\pi_l) = j_l \quad \forall 1 \le l \le m\Big\} = \frac{1}{m!}\frac{k!}{j_1! \cdots j_m!}.$$

Second, for any $\pi \in \mathcal{E}_m$,

$$\Big\{(x_{\pi_1}, \cdots, x_{\pi_m}); \quad (x_1, \cdots, x_k) \in \mathcal{A}(\pi)\Big\} = \mathbb{Z}^m$$

and the map $(x_1, \cdots, x_k) \mapsto (x_{\pi_1}, \cdots, x_{\pi_m})$ is an 1-1 correspondence between $\mathcal{A}(\pi)$ and $\mathbb{Z}^m$.

Observing that

$$\begin{aligned} &S(i_\ell) = x_\ell \text{ for all } \ell \in \{1, \ldots, k\} \Leftrightarrow (x_1, \ldots, x_k) \in \mathcal{A}(\pi) \\ &\text{and } S(i_\ell^*) = x_{\pi_\ell} \text{ for all } \ell \in \{1, \ldots, m\} \end{aligned}$$

we get that

$$\begin{aligned} &\mathbb{E} \sum_{i_1, \cdots, i_k = 1}^{\infty} \prod_{l=1}^{k} 1_{\{S(i_l) = x_l\}} \\ &= \sum_{m=1}^{k} \sum_{\pi \in \mathcal{E}_m} 1_{\{(x_1, \cdots, x_k) \in \mathcal{A}(\pi)\}} \sum_{\substack{j_1, \ldots, j_m \\ \text{distinct}}} \mathbb{E} \prod_{l=1}^{m} 1_{\{S(j_l) = x_{\pi_l}\}} \\ &= \sum_{m=1}^{k} \sum_{\pi \in \mathcal{E}_m} 1_{\{(x_1, \cdots, x_k) \in \mathcal{A}(\pi)\}} \sum_{\sigma \in \Sigma_m} \prod_{l=1}^{m} \mathfrak{G}(x_{\pi_{\sigma(l)}} - x_{\pi_{\sigma(l-1)}}). \end{aligned}$$

Here, and elsewhere in the rest of this section, we follow the convention that $x_{\pi_{\sigma(0)}} = 0$.

Therefore,

$$\mathbb{E}I_\infty^k = \sum_{x_1,\cdots,x_k\in\mathbb{Z}^d}\bigg[\sum_{m=1}^k\sum_{\pi\in\mathcal{E}_m} 1_{\{(x_1,\cdots,x_k)\in\mathcal{A}(\pi)\}} \times\sum_{\sigma\in\Sigma_m}\prod_{l=1}^m \mathfrak{G}(x_{\pi_{\sigma(l)}}-x_{\pi_{\sigma(l-1)}})\bigg]^p. \tag{7.5.10}$$

By (7.5.1) in Theorem 7.5.1, the next lemma gives the lower bound of (7.5.9).

Lemma 7.5.4 *If there exists $h\ge 0$ with $\|h\|_q\le\theta$ such that $\|\mathfrak{K}_h\|>1$, then*

$$\sum_{k=1}^\infty \frac{\theta^k}{k!}\big[\mathbb{E}I_\infty^k\big]^{\frac{1}{p}} = \infty.$$

Proof. Let $h\ge 0$ with $\|h\|_q\le\theta$. Applying Hölder's inequality to (7.5.10),

$$\begin{aligned}
&\theta^k\big[I_\infty^k\big]^{\frac{1}{p}}\\
&\ge \sum_{x_1,\dots,x_k\in\mathbb{Z}^d}\Big(\prod_{j=1}^k h(x_j)\Big)\sum_{m=1}^k\sum_{\pi\in\mathcal{E}_m}1_{\{(x_1,\dots,x_k)\in\mathcal{A}(\pi)\}}\\
&\qquad\times\sum_{\sigma\in\Sigma_m}\prod_{\ell=1}^m\mathfrak{G}(x_{\pi_{\sigma(\ell)}}-x_{\pi_{\sigma(\ell-1)}})\\
&=\sum_{m=1}^k\sum_{\sigma\in\Sigma_m}\sum_{\pi\in\mathcal{E}_m}\sum_{x_1,\dots,x_m\in\mathbb{Z}^d}\Big(\prod_{\ell=1}^m h(x_\ell)^{\#(\pi_\ell)}\Big)\Big(\prod_{\ell=1}^m\mathfrak{G}(x_{\sigma(\ell)}-x_{\sigma(\ell-1)})\Big),
\end{aligned}$$

where the last step follows from the one-to-one correspondence between $\mathcal{A}(\pi)$ and $\mathbb{Z}^m$. Therefore,

$$\begin{aligned}
\theta^k\big[I_\infty^k\big]^{\frac{1}{p}} &\ge \sum_{m=1}^k m!\sum_{x_1,\dots,x_m\in\mathbb{Z}^d}\sum_{\pi\in\mathcal{E}_m}\Big(\prod_{\ell=1}^m h(x_\ell)^{\#(\pi_\ell)}\Big)\Big(\prod_{\ell=1}^m\mathfrak{G}(x_\ell-x_{\ell-1})\Big)\\
&=\sum_{m=1}^k\sum_{x_1,\dots,x_m\in\mathbb{Z}^d}\sum_{\substack{j_1+\cdots+j_m=k\\ j_1,\cdots,j_m\ge 1}}\frac{k!}{j_1!\cdots j_m!}\Big(\prod_{\ell=1}^m h(x_\ell)^{j_\ell}\Big)\Big(\prod_{\ell=1}^m\mathfrak{G}(x_\ell-x_{\ell-1})\Big).
\end{aligned}$$

Summing up both sides,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\theta^k}{k!} \big[I_\infty^k\big]^{\frac{1}{p}} \\
&\geq \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} \sum_{x_1,\dots,x_m} \Bigg\{ \sum_{\substack{j_1+\cdots+j_m=k \\ j_1,\dots,j_m \geq 1}} \frac{1}{j_1!\cdots j_m!} \prod_{\ell=1}^{m} h(x_\ell)^{j_\ell} \Bigg\} \prod_{\ell=1}^{m} \mathfrak{G}(x_\ell - x_{\ell-1}) \\
&= \sum_{m=1}^{\infty} \sum_{x_1,\dots,x_m} \Bigg\{ \prod_{\ell=1}^{m} \sum_{j=1}^{\infty} \frac{h(x_\ell)^j}{j!} \Bigg\} \prod_{\ell=1}^{m} \mathfrak{G}(x_\ell - x_{\ell-1}) \\
&= \sum_{m=1}^{\infty} \sum_{x_1,\dots,x_m} \prod_{\ell=1}^{m} \big(e^{h(x_\ell)} - 1\big)\, \mathfrak{G}(x_\ell - x_{\ell-1}) .
\end{aligned}$$

By the fact that $\|\mathfrak{K}_h\| > 1$ we can pick $g \geq 0$ with $\|g\|_2 = 1$ such that $\langle g, \mathfrak{K}_h g \rangle > 1$. By monotone convergence we may find a finite set $A \subset \mathbb{Z}^d$ such that

$$\Big\langle g1_A, \mathfrak{K}_h\big(g1_A\big) \Big\rangle > 1$$

and additionally, but without loss of generality, $g, h > 0$ on A. Using spectral theory in the same way as in the proof of Theorem 3.1.1, we can prove that

$$\begin{aligned}
&\liminf_{m\to\infty} \frac{1}{m} \log \sum_{x_1,\dots,x_m} \prod_{\ell=1}^{m} \big(e^{h(x_\ell)} - 1\big)\, \mathfrak{G}(x_\ell - x_{\ell-1}) \\
&\geq p \log \Big\langle g1_A, \mathfrak{K}_h\big(g1_A\big) \Big\rangle > 1.
\end{aligned}$$

This leads to the desired conclusion. □

It remains to prove the upper bound of (7.5.9). Our starting point is the formula (7.5.10) and we first project the problem from the full lattice space $\mathbb{Z}^d$ onto a finite set. We fix a large integer N and write $A_N := [-N, N)^d$ for a box of lattice points. Then every $x \in \mathbb{Z}^d$ can uniquely be written as $x = 2Nz + y$ for $z \in \mathbb{Z}^d$ and $y \in A_N$. Hence, using Hölder's inequality in the second step,

$$\begin{aligned}
\mathbb{E}\big[I_\infty^k\big] &= \sum_{y_1,\dots,y_k\in A_N}\sum_{z_1,\dots,z_k\in\mathbb{Z}^d}\bigg[\sum_{m=1}^k\sum_{\pi\in\mathcal{E}_m} 1_{\{(2Nz_1+y_1,\dots,2Nz_k+x_k)\in\mathcal{A}(\pi)\}}\\
&\quad\times\sum_{\sigma\in\Sigma_m}\prod_{\ell=1}^m \mathfrak{G}\big(2N(z_{\pi_{\sigma(\ell)}}-z_{\pi_{\sigma(\ell-1)}})+(x_{\pi_{\sigma(\ell)}}-x_{\pi_{\sigma(\ell-1)}})\big)\bigg]^p\\
&\le \sum_{y_1,\dots,y_k\in A_N}\bigg[\sum_{m=1}^k\sum_{\pi\in\mathcal{E}_m}\sum_{\sigma\in\Sigma_m}\bigg\{\sum_{z_1,\dots,z_k\in\mathbb{Z}^d} 1_{\{(2Nz_1+y_1,\dots,2Nz_k+y_k)\in\mathcal{A}(\pi)\}}\\
&\quad\times\prod_{\ell=1}^m \mathfrak{G}^p\big(2N(z_{\pi_{\sigma(\ell)}}-z_{\pi_{\sigma(\ell-1)}})+(y_{\pi_{\sigma(\ell)}}-y_{\pi_{\sigma(\ell-1)}})\big)\bigg\}^{1/p}\bigg]^p.
\end{aligned}$$

From the uniqueness of the decomposition $x = 2Nz + y$ we infer that

$$1_{\{(2Nz_1+y_1,\dots,2Nz_k+y_k)\in\mathcal{A}(\pi)\}} = 1_{\{(y_1,\dots,y_k)\in\mathcal{A}(\pi)\}}\,1_{\{(z_1,\dots,z_k)\in\mathcal{A}(\pi)\}}.$$

Therefore,

$$\begin{aligned}
&\sum_{z_1,\dots,z_k\in\mathbb{Z}^d} 1_{\{(2Nz_1+y_1,\dots,2Nz_k+y_k)\in\mathcal{A}(\pi)\}}\\
&\times\prod_{\ell=1}^m \mathfrak{G}^p\big(2N(z_{\pi_{\sigma(\ell)}}-z_{\pi_{\sigma(\ell-1)}})+(y_{\pi_{\sigma(\ell)}}-y_{\pi_{\sigma(\ell-1)}})\big)\\
&= 1_{\{(y_1,\dots,y_k)\in\mathcal{A}(\pi)\}}\sum_{z_1,\dots,z_k\in\mathbb{Z}^d} 1_{\{(z_1,\dots,z_k)\in\mathcal{A}(\pi)\}}\\
&\times\prod_{\ell=1}^m \mathfrak{G}^p\big(2N(z_{\pi_{\sigma(\ell)}}-z_{\pi_{\sigma(\ell-1)}})+(y_{\pi_{\sigma(\ell)}}-y_{\pi_{\sigma(\ell-1)}})\big)\\
&= 1_{\{(y_1,\dots,y_k)\in\mathcal{A}(\pi)\}}\sum_{z_1,\dots,z_m\in\mathbb{Z}^d}\\
&\times\prod_{\ell=1}^m \mathfrak{G}^p\big(2N(z_{\sigma(\ell)}-z_{\sigma(\ell-1)})+(y_{\pi_{\sigma(\ell)}}-y_{\pi_{\sigma(\ell-1)}})\big)
\end{aligned}$$

where the last step follows again from one-to-one correspondence between $\mathcal{A}(\pi)$ and $\mathbb{Z}^m$. By permuting the index, the right hand side is equal to

$$1_{\{(y_1,\dots,y_k)\in\mathcal{A}(\pi)\}}\prod_{\ell=1}^m\widetilde{\mathfrak{G}}^p(y_{\pi_{\sigma(\ell)}}-y_{\pi_{\sigma(\ell-1)}})$$

where

$$\widetilde{\mathfrak{G}}(x) = \bigg\{\sum_{z\in\mathbb{Z}^d}\mathfrak{G}^p(2Nz+x)\bigg\}^{1/p}.$$

Summarizing our argument, we have

$$\mathbb{E}\big[I_\infty^k\big] \le \sum_{y_1,\dots,y_k\in A_N}\Big[\sum_{m=1}^k\sum_{\pi\in\mathcal{E}_m} 1_{\{(y_1,\dots,y_k)\in\mathcal{A}(\pi)\}} \times \sum_{\sigma\in\Sigma_m}\prod_{\ell=1}^m \widetilde{\mathfrak{G}}(y_{\pi_{\sigma(\ell)}} - y_{\pi_{\sigma(\ell-1)}})\Big]^p. \tag{7.5.11}$$

For each $h \in \mathcal{L}^q(A_N)$, let $\widetilde{\mathfrak{K}}_h$ be the bounded, self-adjoint operator on $\mathcal{L}^2(A_N)$ define as

$$\widetilde{\mathfrak{K}}_h g(x) = \sqrt{e^{h(x)}-1}\sum_{y\in A_N}\widetilde{\mathfrak{G}}(x-y)g(y)\sqrt{e^{h(y)}-1} \qquad g\in\mathcal{L}^2(A_N).$$

Write

$$\mathcal{C}_N(S) = \inf\Big\{||h||_q;\quad h\ge 0 \text{ and } ||\widetilde{\mathfrak{K}}_h||\ge 1\Big\}$$

and

$$H_k = \sum_{y_1,\dots,y_k\in A_N}\Big[\sum_{m=1}^k\sum_{\pi\in\mathcal{E}_m} 1_{\{(y_1,\dots,y_k)\in\mathcal{A}(\pi)\}}\sum_{\sigma\in\Sigma_m}\prod_{\ell=1}^m \widetilde{\mathfrak{G}}(y_{\pi_{\sigma(\ell)}} - y_{\pi_{\sigma(\ell-1)}})\Big]^p.$$

Lemma 7.5.5

$$\limsup_{k\to\infty}\frac{1}{k}\log\frac{1}{(k!)^p}H_k \le -p\log\mathcal{C}_N(S).$$

Proof. The following combinatorial estimate is partially similar to the one used in the proof of Theorem 3.1.2. Fix a vector $\mathbf{x} = (x_1,\dots,x_k)$ of length k with entries from the set A_N and associate its empirical measure $L_k^{\mathbf{x}}$ by letting

$$L_k^{\mathbf{x}} = \frac{1}{k}\sum_{j=1}^k \delta_{x_j}\,.$$

For each $\tau\in\Sigma_k$ and $\pi\in\mathcal{E}_m$ we denote by $\tau(\pi)\in\mathcal{E}_m$ the partition consisting of the sets $\tau(\pi)_\ell := \tau(\pi_\ell)$ for $\ell\in\{1\dots,m\}$. Then, for any $\tau\in\Sigma_k$ and

$\mathbf{x} = (x_1, \dots, x_k)$, we get

$$\begin{aligned}
&\sum_{m=1}^{k} \sum_{\pi \in \mathcal{E}_m} 1_{\{(x_1,\dots,x_k)\in \mathcal{A}(\pi)\}} \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(x_{\pi_{\sigma(\ell)}} - x_{\pi_{\sigma(\ell-1)}}) \\
&= \sum_{y_1,\dots,y_k\in A_N} 1_{\{L_k^{\mathbf{y}}=L_k^{\mathbf{x}}\}} 1_{\{\mathbf{x}=\mathbf{y}\circ\tau\}} \sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} 1_{\{(y_1,\dots,y_k)\in\mathcal{A}(\tau(\pi))\}} \\
&\qquad \times \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_{\tau(\pi_{\sigma(\ell)})} - y_{\tau(\pi_{\sigma(\ell-1)})}) \\
&= \sum_{y_1,\dots,y_k\in A_N} 1_{\{L_k^{\mathbf{y}}=L_k^{\mathbf{x}}\}} 1_{\{\mathbf{x}=\mathbf{y}\circ\tau\}} \sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} 1_{\{(y_1,\dots,y_k)\in\mathcal{A}(\pi)\}} \\
&\qquad \times \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_{\pi_{\sigma(\ell)}} - y_{\pi_{\sigma(\ell-1)}}) \,.
\end{aligned}$$

Observe that, abbreviating $\mu = L_k^{\mathbf{x}}$ and assuming $L_k^{\mathbf{y}} = L_k^{\mathbf{x}}$, we have

$$\sum_{\tau\in\Sigma_k} 1_{\{\mathbf{x}=\mathbf{y}\circ\tau\}} = \prod_{x\in A_N} \big(k\mu(x)\big)! \,,$$

and hence summing the previous expression over all permutations $\tau \in \Sigma_k$ gives

$$\begin{aligned}
&k! \sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} 1_{\{(x_1,\dots,x_k)\in\mathcal{A}(\pi)\}} \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(x_{\pi_{\sigma(\ell)}} - x_{\pi_{\sigma(\ell-1)}}) \\
&= \prod_{x\in A_N} \big(k\mu(x)\big)! \sum_{y_1,\dots,y_k\in A_N} 1_{\{L_k^{\mathbf{y}}=\mu\}} \sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} 1_{\{(y_1,\dots,y_k)\in\mathcal{A}(\pi)\}} \\
&\times \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_{\pi_{\sigma(\ell)}} - y_{\pi_{\sigma(\ell-1)}}) \,.
\end{aligned}$$

Write $\varphi_\mu(x) = \mu(x)^{1/q}$ for all $x \in A_N$ and note that

$$\begin{aligned}
&\sum_{y_1,\ldots,y_k} 1_{\{L_k^{\mathbf{y}}=\mu\}} \sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} 1_{\{(y_1,\ldots,y_k)\in\mathcal{A}_N(\pi)\}} \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_{\pi_{\sigma(\ell)}} - y_{\pi_{\sigma(\ell-1)}}) \\
&\le \Big(\prod_{x\in A_N} \varphi_\mu(x)^{-k\mu(x)} \Big) \sum_{y_1,\ldots,y_k} \varphi_\mu(y_1)\cdots\varphi_\mu(y_k) \\
&\times \sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} 1_{\{(y_1,\ldots,y_k)\in\mathcal{A}(\pi)\}} \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_{\pi_{\sigma(\ell)}} - y_{\pi_{\sigma(\ell-1)}}) \\
&. = \Big(\prod_{x\in A_N} \varphi_\mu(x)^{-k\mu(x)} \Big) \sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} \sum_{y_1,\ldots,y_k} 1_{\{(y_1,\ldots,y_k)\in\mathcal{A}(\pi)\}} \\
&\times \Big\{ \prod_{\ell=1}^{m} \varphi_\mu(y_{\pi_\ell})^{\#(\pi_\ell)} \Big\} \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_{\pi_{\sigma(\ell)}} - y_{\pi_{\sigma(\ell-1)}}).
\end{aligned}$$

By the one-to-one correspondence between $\mathcal{A}(\pi)$ and $\mathbb{Z}^m$, the the summation over m on the right hand side is equal to

$$\begin{aligned}
&\sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} \sum_{y_1,\ldots,y_m} \Big\{ \prod_{\ell=1}^{m} \varphi_\mu(y_\ell)^{\#(\pi_\ell)} \Big\} \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_{\sigma(\ell)} - y_{\sigma(\ell-1)}) \\
&= \sum_{m=1}^{k} m! \sum_{y_1,\ldots,y_m} \Big\{ \sum_{\pi\in\mathcal{E}_m} \prod_{\ell=1}^{m} \varphi_\mu(y_\ell)^{\#(\pi_\ell)} \Big\} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_\ell - y_{\ell-1}) \\
&= \sum_{m=1}^{k} \sum_{y_1,\ldots,y_m} \Big\{ \sum_{\substack{j_1+\cdots+j_m=k \\ j_1,\ldots,j_m\ge 1}} \frac{k!}{j_1!\cdots j_m!} \prod_{\ell=1}^{m} \varphi_\mu(y_\ell)^{j_\ell} \Big\} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_\ell - y_{\ell-1}).
\end{aligned}$$

Further, using Stirling's formula, we obtain a fixed polynomial $P(\cdot)$, depending only on the cardinality of A_N, such that

$$\Big\{ \prod_{x\in A_N} (k\mu(x))! \Big\} \Big\{ \prod_{x\in A_N} \varphi_\mu(x)^{-k\mu(x)} \Big\} \le P(k)\, k! \Big\{ \prod_{j=1}^{k} \mu(x_j)^{\frac{1}{p}} \Big\}.$$

Summarizing what we have got so far

$$\sum_{m=1}^{k} \sum_{\pi\in\mathcal{E}_m} 1_{\{(x_1,\dots,x_k)\in\mathcal{A}(\pi)\}} \sum_{\sigma\in\Sigma_m} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(x_{\pi_{\sigma(\ell)}} - x_{\pi_{\sigma(\ell-1)}})$$
$$\le P(k)\, k! \left\{ \prod_{j=1}^{k} \mu(x_j)^{\frac{1}{p}} \right\} \sum_{m=1}^{k} \sum_{y_1,\dots,y_m\in A_N}$$
$$\times \left\{ \sum_{\substack{j_1+\cdots+j_m=k\\ j_1,\dots,j_m\ge 1}} \frac{1}{j_1!\cdots j_m!} \prod_{\ell=1}^{m} \varphi_\mu^{j_\ell}(y_\ell) \right\} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_\ell - y_{\ell-1})\,.$$

Summing over all possible vectors $\mathbf{x} = (x_1, \dots, x_k)$ gives

$$H_k \le P(k)^p\, (k!)^p \sup_{\nu\in\mathcal{P}_k(A_N)} \Bigg[\sum_{m=1}^{k} \sum_{y_1,\dots,y_m\in A_N}$$
$$\times \Bigg(\sum_{\substack{j_1+\cdots+j_m=k\\ j_1,\dots,j_m\ge 1}} \frac{1}{j_1!\cdots j_m!} \prod_{\ell=1}^{m} \varphi_\nu^{j_\ell}(y_\ell) \Bigg) \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_\ell - y_{\ell-1}) \Bigg]^p,$$

where $\mathcal{P}_k(A_N)$ is the set of probability densities ν on A_N such that $\nu(x)$ is of the form i/k. Recall that $\varphi_\mu \ge 0$ satisfies $\|\varphi_\mu\|_q = 1$ and hence we may let the supremum take over all L^q-normalized non-negative functions f on A_N with $\|f\|_q = 1$. Thus, for every $\theta > 0$,

$$\frac{\theta^k}{k!} H_k^{\frac{1}{p}} \le P(k) \sup_{\|f\|_q=1} \sum_{n=1}^{\infty} \theta^n \sum_{m=1}^{n} \sum_{y_1,\dots,y_m\in A_N}$$
$$\times \Bigg(\sum_{\substack{j_1+\cdots+j_m=k\\ j_1,\dots,j_m\ge 1}} \frac{1}{j_1!\cdots j_m!} \prod_{\ell=1}^{m} f^{j_\ell}(y_\ell) \Bigg) \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_\ell - y_{\ell-1})$$
$$= P(k) \sup_{\|h\|_q=\theta} \sum_{m=1}^{\infty} \sum_{y_1,\dots,y_m\in A_N} \left\{ \prod_{\ell=1}^{m} \sum_{j=1}^{\infty} \frac{h(y_\ell)^j}{j!} \right\} \prod_{\ell=1}^{m} \widetilde{\mathfrak{G}}(y_\ell - y_{\ell-1})$$
$$= P(k) \sup_{\|h\|_q=\theta} \sum_{m=1}^{\infty} \sum_{y_1,\dots,y_m\in A_N} \prod_{\ell=1}^{m} \Big(e^{h(y_\ell)} - 1 \Big) \widetilde{\mathfrak{G}}(y_\ell - y_{\ell-1}).$$

Similar to (3.1.10),

$$\limsup_{m\to\infty} \frac{1}{m} \log \sup_{\|h\|_q=\theta} \sum_{y_1,\dots,y_m\in A_N} \prod_{\ell=1}^{m} \Big(e^{h(y_\ell)} - 1 \Big) \widetilde{\mathfrak{G}}(y_\ell - y_{\ell-1})$$
$$\le \log \sup_{\|h\|_q=\theta} \sup_{||g||_2=1} \Big| \langle g, \widetilde{\mathfrak{K}}_h g \rangle \Big| = \log \sup_{\|h\|_q=\theta} ||\widetilde{\mathfrak{K}}_h||.$$

Suppose that $\theta < \mathcal{C}_N(S)$, then there exists $\epsilon > 0$ such that every h with $\|h\|_q = \theta$ satisfies $\|\widetilde{\mathfrak{K}}_h\| < 1 - \epsilon$. Consequently, there is a constant $C > 0$ such that

$$\frac{\theta^k}{k!} H_k^{\frac{1}{p}} \leq CP(k)$$

and hence

$$\limsup_{k\to\infty} \frac{1}{k} \log \frac{1}{(k!)^p} H_k \leq -p \log \theta \,,$$

as required to complete the proof. □

By (7.5.11) and Lemma 7.5.5, the following lemma leads to the upper bound of (7.5.9) and therefore completes the proof of Theorem 7.5.3.

Lemma 7.5.6

$$\liminf_{N\to\infty} \mathcal{C}_N(S) \geq \mathcal{C}(S).$$

Proof. Fix positive integers $M < N$. We decompose $\mathfrak{G} = \mathfrak{G}_+ + \mathfrak{G}_-$ with $\mathfrak{G}_+(x) = \mathfrak{G}(x) 1_{\{x \notin [M,M)^d\}}$ and $\mathfrak{G}_-(x) = \mathfrak{G}(x) 1_{\{x \in [M,M)^d\}}$. In analogy to $\widetilde{\mathfrak{G}}$ we define symmetric, periodic functions $\widetilde{\mathfrak{G}}_+$ and $\widetilde{\mathfrak{G}}_-$ by

$$\widetilde{\mathfrak{G}}_\pm(y) = \Big\{ \sum_{z\in\mathbb{Z}^d} \mathfrak{G}_\pm^p\big(2Nz + y\big) \Big\}^{\frac{1}{p}} \qquad \text{for any } y \in \mathbb{Z}^d \,.$$

By the triangle inequality we have $\widetilde{\mathfrak{G}}_N \leq \widetilde{\mathfrak{G}}_+ + \widetilde{\mathfrak{G}}_-$. The induced operators $\widetilde{\mathfrak{K}}_{h,\pm} \colon L^2(A_N) \to L^2(A_N)$ are defined by

$$\widetilde{\mathfrak{A}}_{h,\pm} g(x) = \sqrt{e^{h(x)} - 1} \sum_{y\in A_N} \widetilde{\mathfrak{G}}_\pm(x-y) \sqrt{e^{h(y)} - 1}\, g(y) \,.$$

For the norms of these operators we thus obtain

$$\|\widetilde{\mathfrak{K}}_h\| \leq \|\widetilde{\mathfrak{K}}_{h,+}\| + \|\widetilde{\mathfrak{K}}_{h,-}\| \,.$$

Similar to (7.5.7),

$$\|\widetilde{\mathfrak{K}}_{h,+}\| \leq \Big(e^{||h||_q} - 1\Big) \|\widetilde{\mathfrak{G}}_{h,+}\|_p = \Big(e^{||h||_q} - 1\Big) \Big\{ \sum_{x\in\mathbb{Z}^d} \mathfrak{G}_+^p(x) \Big\}^{1/p}$$

$$= \Big(e^{||h||_q} - 1\Big) \Big\{ \sum_{x\notin[-M,M)^d} \mathfrak{G}^p(x) \Big\}^{1/p} .$$

Define

$$E_N = [-N, N)^d \setminus [-M, M)^d.$$

For any non-negative function h on A_N, extend it (without changing the notation) into a function on $\mathbb{Z}^d$ with $h(x) = 0$ for $x \in \mathbb{Z}^d \setminus A_N$. Similar to (3.2.14),

$$\begin{aligned}\|\widetilde{\mathfrak{K}}_{h,-}\| &= \sup_{||g||_2=1} \langle g, \mathfrak{K}_{h,-} g\rangle \\ &\le \|\mathfrak{K}_h\| + 2\Big(e^{||h||_q} - 1\Big)\Big(\frac{|E_N|}{(2N)^d}\Big)^{\frac{p-1}{p}} \Big\{\sum_{x\in\mathbb{Z}^d} \mathfrak{G}^p(x)\Big\}^{1/p}.\end{aligned}$$

Summarizing our argument,

$$\begin{aligned}\|\widetilde{\mathfrak{K}}_h\| \le \|\mathfrak{K}_h\| + \Big(e^{||h||_q} - 1\Big)\Bigg(&\Big\{\sum_{x\notin[-M,M)^d} \mathfrak{G}^p(x)\Big\}^{1/p} \\ &+ 2\Big(\frac{|E_N|}{(2N)^d}\Big)^{\frac{p-1}{p}} \Big\{\sum_{x\in\mathbb{Z}^d} \mathfrak{G}^p(x)\Big\}^{1/p}\Bigg).\end{aligned} \tag{7.5.12}$$

We now complete the proof: First, we may assume that the

$$\liminf_{N\to\infty} \mathcal{C}_N(S) < \infty.$$

Hence, there is bounded subsequents of $\{\mathcal{C}_N(S)\}$. For simplicity of notation we may assume that sub sequence is $\{\mathcal{C}_N(S)\}$. For each N, we pick up a non-negative function h_N supported on A_N such that $\|\widetilde{\mathfrak{K}}_{h_N}\| \ge 1$ and $||h_N||_q \le \mathcal{C}_N(S) + N^{-1}$. Take $M = \sqrt{N}$ in (7.5.12) and notice that $\{||h_N||_q\}$ is bounded. From (7.5.12) we have $\|\mathfrak{K}_{h_N}\| \ge 1 - o(1)$ as $N \to \infty$.

For any $\theta > 0$, define

$$\mathcal{C}(S, \theta) = \inf\Big\{||h||_q; \quad h \ge 0 \text{ and } ||\mathfrak{K}_h|| \ge \theta\Big\}$$

(So $\mathcal{C}(S, 1) = \mathcal{C}(S)$). By our argument given above,

$$\liminf_{N\to\infty} \mathcal{C}_N(S) \ge \mathcal{C}(S, 1-\epsilon) \qquad \forall \epsilon > 0. \tag{7.5.13}$$

On the other hand, by the inequality[5]

$$\frac{1}{1-\epsilon}\Big(e^{\theta} - 1\Big) \le e^{\frac{\theta}{1-\epsilon}} - 1$$

[5]This can be proved by comparing Taylor expansion in θ of the both sides.

and by the definition of the operator $\mathfrak{K}_h$, one can see that

$$||\mathfrak{K}_{(1-\epsilon)^{-1}h}|| \geq (1-\epsilon)^{-1}||\mathfrak{K}_h||$$

which implies that $\mathcal{C}(S, 1-\epsilon) \geq (1-\epsilon)\mathcal{C}(S)$. Bringing this back to (7.5.13) and letting $\epsilon \to 0^+$ we have proved Lemma 7.5.6. □

The story of J_∞ is quite different. It is shown in Khanin, Mazel, Shlosman and Sinai ([101]) that in the special case $p = 2$, $d \geq 5$,

$$\exp\Big\{ - t^{\frac{d-2}{d}+\delta}\Big\} \leq \mathbb{P}\{J_\infty \geq t\} \leq \exp\Big\{ - t^{\frac{d-2}{d}-\delta}\Big\} \tag{7.5.14}$$

for large $t > 0$, where $\delta > 0$ can be any previously given number.

The estimate made in (7.5.14) exhibits a behavior pattern drastically different from the one stated in Theorem 7.5.1 and Theorem 7.5.3. The discovery of Khanin *et al* came as a surprise for there was a belief that in the high dimensions, the intersection of independent ranges might behave like a constant multiple of the intersection local time.[6] Here is an explanation for that. For I_∞, the "optimal strategies" to be large is to let the random walks stay inside a big but fixed ball, and repeatedly intersect at same sites. While for J_∞ to be large, the random walks have to intersect at a large amount of different sites. Consequently, the random walks behave in a diffusive way.

Providing a sharp estimate for the tail probability of J_∞ is a challenging task, and the problem still remains open.

Conjecture 7.5.7 *When $p(d-2) > d$, there is a $\widetilde{\mathcal{C}}(d,p) > 0$ such that*

$$\lim_{t\to\infty} t^{-\frac{d-2}{d}} \log \mathbb{P}\big\{J_\infty \geq t\big\} = -\widetilde{\mathcal{C}}(d,p).$$

See the "Notes and comments" section for the background of the conjecture. If the conjecture turns out to be right, it is of great interest to identify the constant $\widetilde{\mathcal{C}}(d,p)$.

7.6 Notes and comments

Section 7.1

Theorem 7.1.1 and Theorem 7.1.2 were first established in [24] and [25], respectively. They are obviously linked to Theorem 4.1.6. The correspondent upper bounds are very likely.

[6]One may want to compare Theorem 7.2.1 and Theorem 7.2.2 in the case $d = 3$ and $p = 2$.

Section 7.2

There does not seem to be an agreement in literature on the definition of moderate deviations. Here we view the moderate deviations as a kind of large deviations governed by some weak laws. In some other works such as [10], on the other hand, the words "moderate deviations" refer to the deviations away from the mean that are smaller than the scale of the mean.

Theorem 7.2.1 and Theorem 7.2.2 were obtained in [24] and [25], respectively. An improvement made here is the optimization of the condition on b_n in Theorem 7.2.1, thanks to Lemma 7.1.4.

A careful reader may notice that the case $d = 1$ is missing from Theorem 7.2.2. The reason behind is that J_n has has a drastically different behavior in the dimension $d = 1$. In the light of weak convergence given Theorem 5.2.4, a large deviation of Gaussian tail is expected when $d = 1$. Indeed, it was proved in [27] that for any $p \geq 1$ ($J_n = \#\big\{S[1,n]\big\}$ when $p = 1$), and for any $b_n \to \infty$ with $b_n = o(n)$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{J_n \geq \lambda\sqrt{nb_n}\big\} = -\frac{p\lambda^2}{2\sigma^2} \qquad (\lambda > 0).$$

Exercise 1. Complete the proof of Theorem 7.2.2.

We now compare Theorem 7.2.2 to the work by van den Berg, Bolthausen and den Hollander ([11]) on the large deviations for the volume of *Wiener sausage* intersection. For $\epsilon > 0$, let $W^\epsilon(t)$ be the ϵ-neighborhood of the Brownian path $W(s)$ up to time t. Using Donsker-Varadhan large deviation for Wiener empirical measures ([52]), den Berg, Bolthausen and den Hollander were able to establish the large deviations for the volume

$$|W^\epsilon(t) \cap \widetilde{W}^\epsilon(t)|$$

of the intersection of two independent Wiener sausages $W^\epsilon(t)$ and $\widetilde{W}^\epsilon(t)$. They proved that for any $c > 0$

$$\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}\Big\{|W^\epsilon(ct) \cap \widetilde{W}^\epsilon(ct)| \geq \frac{t}{\log t}\Big\} = -I_2(c) \tag{7.6.1}$$

as $d = 2$; and

$$\lim_{t\to\infty} t^{-\frac{d-2}{d}} \log \mathbb{P}\Big\{|W^\epsilon(ct) \cap \widetilde{W}^\epsilon(ct)| \geq t\Big\} = -I_d(c) \tag{7.6.2}$$

as $d \geq 3$, where $I_d(c)$ is given in the form of variation. Further, den Berg, Bolthausen and den Hollander pointed out that

$$\lim_{c\to\infty} c^{\frac{4-d}{d}} I_d(c) = \begin{cases} (2\pi)^{-2}\mu_d & d=2 \\ \theta_d(\epsilon)^{-4/d}\mu_d & d=3,4 \end{cases} \tag{7.6.3}$$

where

$$\theta_d(\epsilon) = \epsilon^{d-2} 2\pi^{d/2}/\Gamma\Big(\frac{d-2}{2}\Big), \tag{7.6.4}$$

$$\mu_d = \begin{cases} \inf\Big\{||\nabla\psi||_2^2;\quad \psi\in W^{1.2}(\mathbb{R}^d),\ ||\psi||_2=1,\ \psi||_4=1\Big\} & d=2,3 \\ \inf\Big\{||\nabla\psi||_2^2;\quad \psi\in D(\mathbb{R}^4),\ ||\psi||_4=1\Big\} & d=4, \end{cases}$$

$$W^{1,2}(\mathbb{R}^d) = \big\{f\in\mathcal{L}^2(\mathbb{R}^d);\quad \nabla f\in\mathcal{L}^2(\mathbb{R}^d)\big\},$$

and

$$D(\mathbb{R}^d) = \big\{f\ge 0;\quad \nabla f\in\mathcal{L}^2(\mathbb{R}^d)\big\}.$$

It is not hard to see that $\mu_d = \kappa(3,2)^{-8/d}$ for $d\le 4$. On the other hand, the Brownian scaling implies that

$$|W^\epsilon(ct)\cap\widetilde{W}^\epsilon(ct)| \overset{d}{=} c^{d/2}|W^{\epsilon/\sqrt{c}}(t)\cap\widetilde{W}^{\epsilon/\sqrt{c}}(t)|$$

In the case $d=2$, above discussion leads to

$$\lim_{c\to\infty}\lim_{t\to\infty}\frac{c}{\log t}\log\mathbb{P}\Big\{|W^{\epsilon/\sqrt{c}}(t)\cap\widetilde{W}^{\epsilon/\sqrt{c}}(t)|\ge\frac{t}{c\log t}\Big\}$$
$$= -(2\pi)^{-2}\kappa(2,2)^{-4}$$

This is related to (7.2.5) in Theorem 7.2.2 with $p=2$, where b_n corresponds to $c^{-1}\log t$.

In the case $d=3$,

$$\log\mathbb{P}\Big\{|W^{\epsilon/\sqrt{c}}(t)\cap\widetilde{W}^{\epsilon/\sqrt{c}}(t)|\ge c^{-3/2}t\Big\}$$
$$\sim -\Big(\frac{t^{1/3}}{c}\Big)\theta_3(\epsilon/\sqrt{c})^{-4/3}\kappa(3,2)^{-8/3}$$

as the limit is taken in the order $t\to\infty$, $c\to\infty$. Clearly, this is related to (7.2.6) in Theorem 7.2.2 with b_n corresponding to $c^{-1}t^{1/3}$.

Section 7.4.

Part of the support for Conjecture 7.4.2 comes from van den Berg, Bolthausen and den Hollander's work ([11]). Taking $d = 4$ in (7.6.2),

$$\log \mathbb{P}\Big\{|W^{\epsilon/\sqrt{c}}(t) \cap \widetilde{W}^{\epsilon/\sqrt{c}}(t)| \ge c^{-2}t\Big\}$$
$$\sim -\Big(\frac{\sqrt{t}}{c}\Big)\theta_4(\epsilon/\sqrt{c})^{-1}\kappa(4,2)^{-2}$$

in the limiting order $t \to \infty$, $c \to \infty$. This is a version of (7.4.8) with b_n corresponding to $\sqrt{t}/c$.

Section 7.5

The tail probabilities of I_∞ and J_∞ in the special case $d \ge 5$ and $p = 2$ were first investigated by Khanin, Mazal, Sholosman and Sinai ([101]). Theorem 7.5.3 responses to the demand on the sharp tail asymptotics for I_∞, a problem left by Khanin, Mazal, Sholosman and Sinai. The following exercise contains a partial solution to Conjecture 7.5.7, provided by Khanin, Mazal, Sholosman and Sinai ([101]).

Exercise 2. Assume that $p(d-2) > d$.

(1). Prove that for any $k \ge 1$,

$$\mathbb{E}J_\infty^k = \sum_{m=1}^k \frac{1}{m!}\Big\{\sum_{\substack{j_1+\cdots+j_m=k \\ j_1,\cdots,j_m\ge 1}} \frac{k!}{j_1!\cdots j_m!}\Big\}$$
$$\times \sum_{\substack{x_1,\cdots,x_m\in\mathbb{Z}^d \\ distinct}} \Big[\mathbb{P}\big\{\{x_1,\cdots,x_m\}\subset S[1,\infty)\big\}\Big]^p.$$

Hint: Read the proof of (7.5.10).

(2). Prove that for any $\epsilon > 0$ with

$$\frac{d}{d-2} + \epsilon < p$$

and for each $m \ge 1$,

$$\sum_{\substack{x_1,\cdots,x_m\in\mathbb{Z}^d \\ distinct}} \Big[\mathbb{P}\big\{\{x_1,\cdots,x_m\}\subset S[1,\infty)\big\}\Big]^p$$
$$\le \gamma_S^m \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_m}\prod_{l=1}^m \mathfrak{G}(x_{\sigma(l)} - x_{\sigma(l-1)})\Big]^{\frac{d}{d-2}+\epsilon}.$$

Hint: You need three facts:

$$\mathbb{P}\big\{\{x_1,\cdots,x_m\}\subset S[1,\infty)\big\}\le 1$$

$$\mathbb{P}\{T_{y_1}<\cdots<T_{y_m}<\infty\}\le\prod_{l=1}^m\mathbb{P}\{T_{y_l-y_{l-1}}<\infty\}$$

and

$$\mathbb{P}\{T_x<\infty\}=\gamma_S\mathfrak{G}(x).$$

(3). Establish the Khanin-Mazal-Sholosman-Sinai's upper bound: For any $\delta>0$,

$$\mathbb{P}\{J_\infty\ge t\}\le\exp\Big\{-t^{\frac{d-2}{d}-\delta}\Big\}$$

when t is sufficiently large.

Perhaps the most important progress since the paper Khanin *et al* was made by van den Berg, Bolthausen and den Hollander ([11]). To see this, we take $d\ge 5$ in (7.6.2). van den Berg, Bolthausen and den Hollander ([11]) pointed out that there is a $c_d>0$ such that

$$I_d(c)=\theta_d(\epsilon)^{-1}\inf\bigg\{||\psi||_2^2;\quad \psi\in D(\mathbb{R}^d),\ \int_{\mathbb{R}^d}\big[1-e^{-\psi}\big]^2dx=1\bigg\}$$

for $c\ge c_d$. Consequently,

$$\lim_{t\to\infty}t^{-\frac{d-2}{d}}\log\mathbb{P}\Big\{|W^\epsilon(ct)\cap\widetilde{W}^\epsilon(ct)|\ge t\Big\}=-I_d(c_d)$$

as $c\ge c_d$. Based on this, van den Berg, Bolthausen and den Hollander ([11]) conjectured that

$$\lim_{t\to\infty}t^{-\frac{d-2}{d}}\log\mathbb{P}\Big\{|W^\epsilon(\infty)\cap\widetilde{W}^\epsilon(\infty)|\ge t\Big\}=-I_d(c_d).$$

There is a noticeable similarity between this conjecture and Conjecture 7.5.7.

Chapter 8

Large deviations: single random walk

In this chapter we establish the large deviations for self-intersection local time and range of a single d-dimensional random walk. Comparing with the previous chapter, our main theorems in this chapter is more dimension-dependent. Indeed, the results in the dimensions $d = 1$ and $d = 2$ appears as the moderate deviations in connection to the large deviations given in Theorem 4.2.1, Theorem 4.3.1 and Theorem 4.3.2. When $d = 3$ and when $d = 4$, the Brownian self-intersection local times can not be defined even under renormalization. On the other hand, the Galiardo-Nirenberg inequality (when $d = 3$) and Sobolev inequality (when $d = 4$) remains valid. This suggests that the type of large deviations achieved in Cheaper 4 may be extended to higher dimensions in some deviation scales larger than the moderate deviations. Finally, lack of Galiardo-Nirenberg inequality or Sobolev inequality makes the dimensions $d \geq 5$ very different from the case $d \leq 4$.

8.1 Self-intersection in one dimension

Let $S(n)$ be a symmetric random walk on $\mathbb{Z}$ with variance $\sigma^2 > 0$. Assume that the smallest sub-group of $\mathbb{Z}$ that supports $S(n)$ is $\mathbb{Z}$. Recall that $l(n, x)$ is the local time of $S(n)$ defined in (5.2.1). By (5.2.2) and(5.2.3), the study of the large deviations for the p-multiple self-intersection local time is equivalent to the study of the large deviations for the p-square of the local time given by

$$\sum_{x\in\mathbb{Z}} l^p(n,x) \qquad n=1,2,\cdots.$$

The following is the main theorem of this section.

Theorem 8.1.1 *Let $p>1$ be fixed.*

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\sum_{x\in\mathbb{Z}} l^p(n,x)\ge n^{\frac{p+1}{2}}b_n^{\frac{p-1}{2}}\Big\}$$
$$=-\frac{\sigma^2}{4(p-1)}\Big(\frac{p+1}{2}\Big)^{\frac{3-p}{p-1}}B\Big(\frac{1}{p-1},\frac{1}{2}\Big)$$

for any positive sequence b_n satisfying

$$b_n\longrightarrow\infty \text{ and } b_n=o(n) \qquad (n\to\infty)$$

where $B(\cdot,\cdot)$ is the beta function.

Proof. By Corollary 1.2.4 and by Theorem C.1 in Appendix, it suffices to show that for any $\theta>0$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p}\Big\} \tag{8.1.1}$$
$$=\sup_{g\in\mathcal{F}_1}\Big\{\sigma^{-\frac{p-1}{p}}\theta\Big(\int_{-\infty}^{\infty}|g(x)|^{2p}dx\Big)^{1/p}-\int_{-\infty}^{\infty}|g'(x)|^2dx\Big\}.$$

Indeed, by variable substitution,

$$\sum_{x\in\mathbb{Z}} l^p(n,x)=\int_{-\infty}^{\infty}l^p(n,[x])dx=\sqrt{\frac{n}{b_n}}\int_{-\infty}^{\infty}l^p\Big(n,\Big[\sqrt{\frac{n}{b_n}}x\Big]\Big)dx.$$

Let $f(x)$ be a bounded, uniformly continuous function on $(-\infty,\infty)$ such that $||f||_q=1$. By Hólder inequality,

$$\begin{aligned}\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p}&\ge\Big(\frac{n}{b_n}\Big)^{\frac{1}{2p}}\int_{-\infty}^{\infty}f(x)l\Big(n,\Big[\sqrt{\frac{n}{b_n}}x\Big]\Big)dx\\&=\Big(\frac{b_n}{n}\Big)^{\frac{p-1}{2p}}\int_{-\infty}^{\infty}f\Big(\sqrt{\frac{b_n}{n}}x\Big)l(n,[x])dx\\&=\Big(\frac{b_n}{n}\Big)^{\frac{p-1}{2p}}\Big\{o(n)+\sum_{x\in\mathbb{Z}^d}f\Big(\sqrt{\frac{b_n}{n}}x\Big)l(n,x)\Big\}\\&=\Big(\frac{b_n}{n}\Big)^{\frac{p-1}{2p}}\Big\{o(n)+\sum_{k=1}^{n}f\Big(\sqrt{\frac{b_n}{n}}S(k)\Big)\Big\}.\end{aligned}$$

By Feynman-Kac minoration given in Theorem 7.1.1,

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}}l^p(n,x)\Big)^{1/p}\right\}$$
$$\geq\sup_{g\in\mathcal{F}_1}\left\{\theta\int_{-\infty}^{\infty}f(\sigma x)g^2(x)dx-\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.$$

Taking supremum over f leads to the lower bound

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}}l^p(n,x)\Big)^{1/p}\right\} \tag{8.1.2}$$
$$\geq\sup_{g\in\mathcal{F}_1}\left\{\sigma^{-\frac{p-1}{p}}\theta\Big(\int_{-\infty}^{\infty}|g(x)|^{2p}dx\Big)^{1/p}-\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.$$

On the other hand, let $t>0$ be fixed for a moment and write $t_n=[tn/b_n]$, $\gamma_n=[n/b_n]$. By sub-additivity,

$$\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}}l^p(n,x)\Big)^{1/p}\right\}$$
$$\leq\left(\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}}l^p(t_n,x)\Big)^{1/p}\right\}\right)^{\gamma_n+1}.$$

By Theorem 5.2.3,

$$\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}}l^p(t_n,x)\Big)^{1/p}\xrightarrow{d}\sigma^{-\frac{p-1}{p}}\Big(\int_{-\infty}^{\infty}L^p(t,x)dx\Big)^{1/p}$$

where $L(t,x)$ is the local time of an 1-dimensional Brownian motion. In addition, the sub-additivity also implies (see the proof of Theorem 6.3.1) that

$$\sup_{n\geq1}\mathbb{E}\exp\left\{\lambda\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}}l^p(t_n,x)\Big)^{1/p}\right\}<\infty\qquad\lambda>0.$$

Therefore,

$$\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}}l^p(t_n,x)\Big)^{1/p}\right\}$$
$$\longrightarrow\mathbb{E}\exp\left\{\theta\sigma^{-\frac{p-1}{p}}\Big(\int_{-\infty}^{\infty}L^p(t,x)dx\Big)^{1/p}\right\}\qquad(n\to\infty).$$

Summarizing our argument,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}}l^p(n,x)\Big)^{1/p}\right\}\\&\le\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\sigma^{-\frac{p-1}{p}}\Big(\int_{-\infty}^{\infty}L^p(t,x)dx\Big)^{1/p}\right\}.\end{aligned}$$

By (4.2.2), letting $t\to\infty$ on the right hand side leads to the upper bound

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{p+1}{2p}}\Big(\sum_{x\in\mathbb{Z}}l^p(n,x)\Big)^{1/p}\right\}\\&\le\sup_{g\in\mathcal{F}_1}\left\{\sigma^{-\frac{p-1}{p}}\theta\Big(\int_{-\infty}^{\infty}|g(x)|^{2p}dx\Big)^{1/p}-\int_{-\infty}^{\infty}|g'(x)|^2dx\right\}.\end{aligned}$$

Combining this with the lower bound given in (8.1.2) gives (8.1.1). □

Some idea used in the proof of (8.1.1) can be partially applied to the multi-dimensional setting. In the rest of the section $S(n)$ is a symmetric, square integrable random walk supported by $\mathbb{Z}^d$ with $d\ge 1$. Let $\epsilon>0$ be fixed and write

$$l(n,x,\epsilon)=\Big(\frac{b_n}{n}\Big)^{d/2}\det(\Gamma)^{-1/2}\sum_{k=1}^n p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(k)-x\big)\Big),\tag{8.1.3}$$

$$R(n,x,\epsilon)=\Big(\frac{b_n}{n}\Big)^{d/2}\det(\Gamma)^{-1/2}\sum_{y\in S[1,n]}p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}(y-x)\Big).\tag{8.1.4}$$

Theorem 8.1.2 *For any $\theta>0$ and $p>1$,*

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}}\Big(\sum_{x\in\mathbb{Z}^d}l^p(n,x,\epsilon)\Big)^{1/p}\right\}\\&=\sup_{g\in\mathcal{F}_d}\left\{\theta\det(\Gamma)^{-\frac{p-1}{2p}}\Big(\int_{\mathbb{R}^d}|g(x,\epsilon)|^{2p}dx\Big)^{1/p}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\end{aligned}$$

for any positive sequence b_n satisfying

$$b_n\longrightarrow\infty\ \text{and}\ b_n=o(n)\qquad(n\to\infty)$$

where

$$g(x,\epsilon)=\left\{\int_{\mathbb{R}^d}g^2(y)p_\epsilon(y-x)dy\right\}^{1/2}.$$

Theorem 8.1.3 *Let $\theta > 0$ and $p \ge 1$.*

(1) When $d = 2$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{1}{p}} \log n \Big(\sum_{x\in\mathbb{Z}^2} R^p(n,x,\epsilon)\Big)^{1/p}\right\}$$
$$= \sup_{g\in\mathcal{F}_2}\left\{2\pi\theta \det(\Gamma)^{\frac{1}{2p}}\Big(\int_{\mathbb{R}^2} |g(x,\epsilon)|^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{\mathbb{R}^2} |\nabla g(x)|^2 dx\right\}$$

for any positive sequence b_n satisfying

$$b_n \longrightarrow \infty \ \textit{and}\ b_n = o\big(\log n\big) \qquad (n\to\infty).$$

(2) When $d \ge 3$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}} \Big(\sum_{x\in\mathbb{Z}^d} R^p(n,x,\epsilon)\Big)^{1/p}\right\}$$
$$= \sup_{g\in\mathcal{F}_d}\left\{\gamma_S\theta \det(\Gamma)^{-\frac{p-1}{2p}}\Big(\int_{\mathbb{R}^d} |g(x,\epsilon)|^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx\right\}$$

for any positive sequence b_n satisfying

$$b_n \longrightarrow \infty \ \textit{and}\ b_n = o\big(n^{1/3}\big) \qquad (n\to\infty).$$

Here we recall that $\gamma_S = \mathbb{P}\{S(k) \ne 0 \ \ \forall n \ge 1\}$.

The approach for the lower bounds in Theorem 8.1.2 and Theorem 8.1.3 are same as the correspondent argument given in the proof of Theorem 8.1.1. For the upper bounds, the only difference is that here we use Theorem 4.2.3 instead of (4.2.2). We omit the detail of the proof here.

8.2 Self-intersection in $d = 2$

In the multi-dimensions, the self-intersection local times we work with are the ones in connection to double self-intersections (i.e., $p = 2$). Recall that

$$Q_n = \sum_{1\le j<k\le n} 1_{\{S(j)=S(k)\}} \qquad n = 1, 2, \cdots$$

where $\{S(k)\}_{k\ge1}$ is a symmetric random walk in $\mathbb{Z}^d$ with covariance matrix Γ, and the smallest sub-group od $\mathbb{Z}^d$ that support the random walk $\{S(k)\}_{k\ge1}$ is

$\mathbb{Z}^d$ itself. Due to the concentration phenomena when $d \geq 2$, we are interested in the renormalized self-intersection local time $Q_n - \mathbb{E}Q_n$.

In this section we let $d = 2$. By Theorem 5.4.2, $n^{-1}(Q_n - \mathbb{E}Q_n)$ weakly converges to $\det(\Gamma)^{-1/2}\gamma\big([0,1]^2_<\big)$. Here we recall that

$$\begin{aligned}\gamma\big([0,t]^2_<\big) = &\iint_{\{0\leq r<s\leq t\}} \delta_0\big(W(r) - W(s)\big)drds\\ &- \mathbb{E}\iint_{\{0\leq r<s\leq t\}} \delta_0\big(W(r) - W(s)\big)drds\end{aligned}$$

is the renormalized self-intersection local time run by the planar Brownian motion $W(t)$. Hence, the tail behavior of type given in Theorem 4.3.1 and Theorem 4.3.2 is naturally expected for $n^{-1}(Q_n - \mathbb{E}Q_n)$. The following theorem is the main result in this section.

Theorem 8.2.1 *For any $\lambda > 0$ and for any positive sequence with*

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o(n) \qquad (n \to \infty) \tag{8.2.1}$$

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{Q_n - \mathbb{E}Q_n \geq \lambda n b_n\Big\} = -\lambda\sqrt{\det(\Gamma)}\kappa(2,2)^{-4} \tag{8.2.2}$$

where $\kappa(2,2) > 0$ is the best constant of the Gagliardo-Nirenberg inequality defined in (4.3.4).

Proof. By (6.3.19) in Theorem 6.3.3 and by Chebyshev inequality, one can see that

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ -(Q_n - \mathbb{E}Q_n) \geq \lambda n b_n\Big\} = -\infty$$

for every $\lambda > 0$. Consequently, (8.2.2) is equivalent to

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{|Q_n - \mathbb{E}Q_n| \geq \lambda n b_n\Big\} = -\lambda\sqrt{\det(\Gamma)}\kappa(2,2)^{-4}.$$

By Corollary 1.2.4 (with $p = 2$), it suffices to show that

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}}|Q_n - \mathbb{E}Q_n|^{1/2}\right\} \\ &= \frac{1}{4}\theta^2 \det(\Gamma)^{-1/2}\kappa(2,2)^4 \qquad \forall\theta > 0.\end{aligned} \tag{8.2.3}$$

According to Theorem 6.3.2, there is a $c > 0$ such that

$$\sup_{n\geq 1} \mathbb{E}\exp\left\{\frac{c}{n}|Q_n - \mathbb{E}Q_n|\right\} < \infty.$$

A standard argument of Chebyshev type shows that there is a constant $C_Q > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta\sqrt{\frac{b_n}{n}}|Q_n - \mathbb{E}Q_n|^{1/2}\right\} \le C_Q\theta^2 \tag{8.2.4}$$

for every sequence b_n satisfying (8.2.1) and for every $\theta > 0$. In the following we strengthen (8.2.3) into (8.2.2).

One of the key ideas is the following decomposition:

$$Q_n = \sum_{j=1}^{2^N} \eta_j + \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \xi_{j,k} \tag{8.2.5}$$

where N is a large but fixed number,

$$\eta_j = \sum_{(j-1)n2^{-N} < i < i' \le jn2^{-N}} 1_{\{S(i)=S(i')\}}$$

$$\xi_{j,k} = \sum_{\substack{(2k-2)n2^{-j} < i \le (2k-1)n2^{-j} \\ (2k-1)n2^{-j} < i' \le (2k)n2^{-j}}} 1_{\{S(i)=S(i')\}}.$$

The decomposition follows a geometric idea similar to the one contained in Figure 2.1 of Section 2.4. Essentially, our approach here is the argument of triangular approximation.

The sequence $\{\eta_j\}_{1\le j\le 2^N}$ is independent with

$$\eta_j \stackrel{d}{=} Q_{n_j} \qquad j = 1, \cdots 2^N \tag{8.2.6}$$

where

$$n_j = \#((j-1)n2^{-N}, jn2^{-N}] = [2^{-N}n] \text{ or } [2^{-N}n] \pm 1.$$

Here and elsewhere, we sometimes use the notation $(a, b]$ for, without notification when it becomes obvious, the set of the integers in that interval.

For each $1 \le j \le N$, $\{\xi_{j,k}\}_{1\le k\le 2^{j-1}}$ is an independent sequence with

$$\xi_{j,k} \stackrel{d}{=} \sum_{i=0}^{n_{j,k}-1} \sum_{i'=1}^{n'_{j,k}} 1_{S(i)=\widetilde{S}(i')\}} \qquad k = 1, \cdots, 2^{j-1} \tag{8.2.7}$$

where $\{\widetilde{S}(k)\}$ is an independent copy of the random walk $\{S(k)\}$ and

$$n_{j,k} = \#((2k-1)n2^{-j},\ (2k)n2^{-j}] = 2^{-j}n \text{ or } 2^{-j}n \pm 1,$$

$$n'_{j,k} = \#((2k)n2^{-j}\ (2k+1)n2^{-j}] = 2^{-j}n, \text{ or } 2^{-j}n \pm 1.$$

Recall that the smoothed local time $\xi(n,x,\epsilon)$ is defined by

$$l(n,x,\epsilon)=\det(\Gamma)^{-1/2}\Big(\frac{b_n}{n}\Big)\sum_{k=1}^n p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(k)-x\big)\Big).$$

By Theorem 8.1.2 (with $d=p=2$), for each $\epsilon>0$

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{x\in\mathbb{Z}^2}l^2(n,x,\epsilon)\Big)^{1/p}\Big\} \qquad (8.2.8)\\ &=\sup_{g\in\mathcal{F}_2}\Big\{\theta\det(\Gamma)^{-1/4}\Big(\int_{\mathbb{R}^2}|g(x,\epsilon)|^4dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla g(x)|^2dx\Big\}\end{aligned}$$

where

$$\mathcal{F}_2=\Big\{g\in\mathcal{L}^2(\mathbb{R}^2);\quad ||g||_2=1\ \text{ and }\ \nabla g\in\mathcal{L}^2(\mathbb{R}^2)\Big\}.$$

In the following we approximate (8.2.3) by (8.2.3). Define

$$\begin{aligned}\xi_{j,k}(\epsilon)=\det(\Gamma)^{-1}\Big(\frac{b_n}{n}\Big)^2&\sum_{x\in\mathbb{Z}^3}\sum_{\substack{(2k-2)n2^{-j}<i\le(2k-1)n2^{-j}\\(2k-1)n2^{-j}<i'\le(2k)n2^{-j}}}\\ &\times p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(i)-x\big)\Big)p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(i')-x\big)\Big)\end{aligned}$$

where $k=1,\cdots,2^{j-1}$ and $j=1,\cdots,N$.

Define

$$\eta_j(\epsilon)=\det(\Gamma)^{-1/2}\frac{b_n}{n}\Big\{\sum_{x\in\mathbb{Z}^3}\Big[\sum_{(j-1)n2^{-N}<i\le jn2^{-N}}p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(i)-x\big)\Big)\Big]^2\Big\}^{1/2}$$

where $j=1,\cdots,2^N$.

Correspondent to (8.2.5) we have that

$$\sum_{x\in\mathbb{Z}^2}l^2(n,x,\epsilon)=\sum_{j=1}^{2^N}\eta_j^2(\epsilon)+2\sum_{j=1}^N\sum_{k=1}^{2^{j-1}}\xi_{j,k}(\epsilon). \qquad (8.2.9)$$

In addition, $\eta_1(\epsilon),\cdots,\eta_{2^N}(\epsilon)$ are independent with

$$\eta_j(\epsilon)\overset{d}{=}\det(\Gamma)^{-1/2}\frac{b_n}{n}\Big\{\sum_{x\in\mathbb{Z}^2}\Big[\sum_{i=1}^{n_j}p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(i)-x\big)\Big)\Big]^2\Big\}^{1/2}. \qquad (8.2.10)$$

For each $1 \le j \le N$, $1 \le k \le 2^{j-1}$

$$\xi_{j,k} - \xi_{j,k}(\epsilon) \overset{d}{=} \sum_{x\in\mathbb{Z}^2} \sum_{\substack{0\le i\le n_{j,k}-1\\ 1\le i'\le n'_{j,k}}} 1_{\{S(i)=\widetilde{S}(i')=x\}} \tag{8.2.11}$$
$$- \det(\Gamma)^{-1}\Big(\frac{b_n}{n}\Big)^2 \sum_{x\in\mathbb{Z}^2} \sum_{\substack{0\le i\le n_{j,k}-1\\ 1\le i'\le n'_{j,k}}} p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(i)-x\big)\Big)$$
$$\times p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(\widetilde{S}(i')-x\big)\Big)$$

where $n_j, n_j, n_{j,k}, n'_{j,k}$ are defined same as above.

By (8.2.4) and (8.2.6) for each $1 \le j \le 2^N$,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta\sqrt{\frac{b_n}{n}}|\eta_j - \mathbb{E}\eta_j|^{1/2}\Big\} \le 2^{-N} C_Q \theta^2.$$

By part (b), Theorem 1.2.2, for any $\theta > 0$,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{j=1}^{2^N}|\eta_j - \mathbb{E}\eta_j|\Big)^{1/2}\Big\} \le 2^{-N} C_Q \theta^2. \tag{8.2.12}$$

By (8.2.8) and (8.2.10),

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta\sqrt{\frac{b_n}{n}}\eta_j(\epsilon)\Big\}$$
$$\le 2^{-N} \sup_{g\in\mathcal{F}_2}\Big\{\theta \det(\Gamma)^{-1/4}\Big(\int_{\mathbb{R}^2}|g(x,\epsilon)|^4 dx\Big)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^2}|\nabla g(x)|^2 dx\Big\}$$
$$\le 2^{-N} \sup_{g\in\mathcal{F}_2}\Big\{\theta \det(\Gamma)^{-1/4}\Big(\int_{\mathbb{R}^2}|g(x)|^4 dx\Big)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^2}|\nabla g(x)|^2 dx\Big\}$$

where the last step follows from Jensen inequality.

By Theorem C.1 in Appendix, for each $\theta > 0$

$$\sup_{g\in\mathcal{F}_2}\Big\{\theta\Big(\int_{\mathbb{R}^2}|g(x)|^4 dx\Big)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^2}|\nabla g(x)|^2 dx\Big\} = 2\kappa(2,2)^4\theta^2. \tag{8.2.13}$$

Hence,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta\sqrt{\frac{b_n}{n}}\eta_j(\epsilon)\Big\} \le 2^{-(N-1)} \det(\Gamma)^{-1/2}\kappa(2,2)^4\theta^2.$$

By part (b), Theorem 1.2.2 again, for any $\theta > 0$,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{j=1}^{2^N} \eta_j(\epsilon)\Big)\right\} \le 2^{-(N-1)} \det(\Gamma)^{-1/2}\kappa(2,2)^4\theta^2. \tag{8.2.14}$$

The bounds in (8.2.12) and (8.2.14) indicate that for large N, it is the second summations in the decompositions (8.2.5) and (8.2.9) that play a dominating role. By (8.2.11) and by the exponential approximation established in Lemma 8.2.2 below,

$$\lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}} |\xi_{j,k} - \xi_{j,k}(\epsilon)|^{1/2}\right\} = 0.$$

For any fixed but arbitrary N and for any $\theta > 0$,

$$\lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}}\Big(\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} |\xi_{j,k} - \xi_{j,k}(\epsilon)|\Big)^{1/2}\right\} = 0. \tag{8.2.15}$$

In addition, by (8.2.7) one can see that

$$\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \mathbb{E}\xi_{j,k} = O(n) \qquad (n\to\infty).$$

Combining this with (8.2.12), (8.2.14) and (8.2.15), a standard argument by Hölder inequality gives

$$\lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}}\Big|(Q_n - \mathbb{E}Q_n) - \frac{1}{2}\sum_{x\in\mathbb{Z}^2} l^2(n,x,\epsilon)\Big|^{1/2}\right\} = 0.$$

This, together with (8.2.6) (with θ being replaced by $\theta/\sqrt{2}$),

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}} |Q_n - \mathbb{E}Q_n|^{1/2}\right\} \\
&= \lim_{\epsilon\to 0^+} \sup_{g\in\mathcal{F}_2} \left\{\frac{\theta}{\sqrt{2}} \det(\Gamma)^{-1/4}\Big(\int_{\mathbb{R}^2} |g(x,\epsilon)|^4 dx\Big)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^2} |\nabla g(x)|^2 dx\right\}.
\end{aligned}$$

Here the existence of the limit (as $\epsilon \to 0^+$) on the right hand side is briefly proved as follows: By the Jensen inequality,

$$\begin{aligned}
&\limsup_{\epsilon\to 0^+}\sup_{g\in\mathcal{F}_2}\left\{\frac{\theta}{\sqrt{2}}\det(\Gamma)^{-1/4}\left(\int_{\mathbb{R}^2}|g(x,\epsilon)|^4dx\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla g(x)|^2dx\right\}\\
&\le\sup_{g\in\mathcal{F}_2}\left\{\frac{\theta}{\sqrt{2}}\det(\Gamma)^{-1/4}\left(\int_{\mathbb{R}^2}|g(x)|^4dx\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla g(x)|^2dx\right\}.
\end{aligned}$$

On the other hand, for any $f\in\mathcal{F}_2$,

$$\begin{aligned}
&\sup_{g\in\mathcal{F}_2}\left\{\frac{\theta}{\sqrt{2}}\det(\Gamma)^{-1/4}\left(\int_{\mathbb{R}^2}|g(x,\epsilon)|^4dx\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla g(x)|^2dx\right\}\\
&\ge\frac{\theta}{\sqrt{2}}\det(\Gamma)^{-1/4}\left(\int_{\mathbb{R}^2}|f(x,\epsilon)|^4dx\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx.
\end{aligned}$$

By Gagliardo-Nirenbrg inequality defined in (4.3.4), $\mathcal{F}_2\subset\mathcal{L}^4(\mathbb{R}^2)$. Applying Lemma 2.2.2 to f^2 we have that

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^2}|f(x,\epsilon)|^4dx=\int_{\mathbb{R}^d}|f(x)|^4dx.$$

Letting $\epsilon\to 0^+$ on the both side of the inequality and then taking supremum over f on the right hand side we have

$$\begin{aligned}
&\liminf_{\epsilon\to 0^+}\sup_{g\in\mathcal{F}_2}\left\{\frac{\theta}{\sqrt{2}}\det(\Gamma)^{-1/4}\left(\int_{\mathbb{R}^2}|g(x,\epsilon)|^4dx\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla g(x)|^2dx\right\}\\
&\ge\sup_{f\in\mathcal{F}_2}\left\{\frac{\theta}{\sqrt{2}}\det(\Gamma)^{-1/4}\left(\int_{\mathbb{R}^2}|f(x)|^4dx\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla f(x)|^2dx\right\}.
\end{aligned}$$

Finally, the desired (8.2.3) follows from (8.2.13). □

As we have seen in above proof, our argument substantially relies on the approximation of the intersection local time of two independent random walks by its smoothed version. Here we provide support to such approach.

Write

$$I_n=\sum_{x\in\mathbb{Z}^2}l(n,x)\widetilde{l}(n,x)$$

and for any $\epsilon>0$, write

$$I_n(\epsilon)=\sum_{x\in\mathbb{Z}^2}l(n,x,\epsilon)\widetilde{l}(n,x,\epsilon)$$

where $l(n,x)$ and $\widetilde{l}(n,x)$ are the local times of the independent 2-dimensional random walks $\{S(k)\}$ and $\{\widetilde{S}(k)\}$, respectively; and $\widetilde{l}(n,x,\epsilon)$ is generated by $\{\widetilde{S}(k)\}$ of $\{S(k)\}$, in the same way as $l(n,x,\epsilon)$ is generated by $\{S(k)\}$.

Lemma 8.2.2 *For any $\theta > 0$*

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}}|I_n - I_n(\epsilon)|^{1/2}\right\} = 0.$$

Proof. By the resolvent approximation performed in the proof of Theorem 7.1.1, we may assume that the random walks $S(k)$ and $\widetilde{S}(k)$ are aperiodic. Define

$$\begin{aligned}\bar{I}_n(\epsilon) = K_{n,\epsilon}^{-2}\sum_{x\in\mathbb{Z}^2}&\left[\sum_{k=1}^n p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(k)-x\big)\Big)\right]\\ &\times\left[\sum_{k=1}^n p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(\widetilde{S}(k)-x\big)\Big)\right]\end{aligned}$$

where

$$K_{n,\epsilon} = \sum_{y\in\mathbb{Z}^2} p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}y\Big) \sim \sqrt{\det(\Gamma)}\frac{n}{b_n} \qquad (n\to\infty).$$

One can see that for each $\epsilon > 0$ there is a constant sequence $C_n \to 1$ such that $I_n(\epsilon) = C_n\bar{I}_n(\epsilon)$.

From (7.2.3) (with $d = p = 2$), one can see that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}}I_n^{1/2}\right\} < \infty \qquad \forall\theta > 0.$$

Consequently, it suffices to show

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}}|I_n - \bar{I}_n(\epsilon)|^{1/2}\right\} = 0. \tag{8.2.16}$$

By Taylor expansion and by Cauchy-Schwartz inequality,

$$\begin{aligned}&\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n}}|I_n - \bar{I}_n(\epsilon)|^{1/2}\right\}\\ &\le \sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{m/2}\Big\{\mathbb{E}|I_n - \bar{I}_n(\epsilon)|^m\Big\}^{1/2}.\end{aligned}$$

By the fact that $\mathbb{E}\big[I_n - \bar{I}_n(\epsilon)\big]^m \geq 0$ for all integers $m \geq 0$, and by the fact that on the right hand side, the terms with odd power m can be dominated properly by the terms with even power through Cauchy-Schwartz inequality, we need only to show that

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{m/2}\Big\{\mathbb{E}\big[I_n - \bar{I}_n(\epsilon)\big]^m\Big\}^{1/2} = 0. \tag{8.2.17}$$

For sufficiently large n, let $0 = n_0 < n_1 < \cdots < n_{[b_n]} = n$ be an integer partition of $[0, n]$ with $n_k - n_{k-1} = [n/b_n]$ or $[n/b_n] + 1$ for each $1 \leq k \leq [b_n]$. Take the additive functional $\mathcal{F}_k^x(S) = l(n_k, x)$ $(k = 1, \cdots, [b_n])$ on the increasing sequence $n_1 < \cdots < n_{[b_n]}$ and the probability density function

$$h(x) = K_{n,\epsilon}^{-1} p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}x\Big)$$

in (6.1.15), Theorem 6.1.2.

$$\begin{aligned}&\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{m/2}\Big\{\mathbb{E}\big[I_n - \bar{I}_n(\epsilon)\big]^m\Big\}^{1/2} \\ &\leq \prod_{k=1}^{[b_n]}\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{m/2}\Big\{\mathbb{E}\big|I_{n_k-n_{k-1}} - \bar{I}_{n_k-n_{k-1},\epsilon}\big|^m\Big\}^{1/2},\end{aligned} \tag{8.2.18}$$

where, in connection to (5.3.1) with $d = p = 2$,

$$\begin{aligned}\bar{I}_{n_k-n_{k-1},\epsilon} &\equiv K_{n,\epsilon}^{-2}\sum_{x\in\mathbb{Z}^2}\Big[\sum_{j=1}^{n_k-n_{k-1}} p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(j)-x\big)\Big)\Big] \\ &\quad\times\Big[\sum_{j=1}^{n_k-n_{k-1}} p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(\widetilde{S}(j)-x\big)\Big)\Big] \\ &= \big(1+o(1)\big)I_{n_k-n_{k-1},\epsilon} \qquad (n\to\infty).\end{aligned}$$

Replacing n by $n_k - n_{k-1}$ in Theorem 5.3.1 (with $d = p = 2$) [1] we have

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\Big(\frac{b_n}{n}\Big)^2\mathbb{E}\Big[I_{n_k-n_{k-1}} - \bar{I}_{n_k-n_{k-1},\epsilon}\Big]^2 = 0.$$

Further, (6.2.1) in Theorem 6.2.1 (with $d = p = 2$) ensures that the dominated convergence theorem applies to our setting so that

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{m/2}\Big\{\mathbb{E}\big|I_{n_k-n_{k-1}} - \bar{I}_{n_k-n_{k-1},\epsilon}\big|^m\Big\}^{1/2} = 1.$$

[1] Here is where we use the extra aperiodicity assumption.

Obviously, the limit holds uniformly for $1 \le k \le [b_n]$. Consequently, the requested (8.2.17) follows from (8.2.18). □

The lower tail asymptotics established in Theorem 4.3.2 can be partially extended to $Q_n - \mathbb{E}Q_n$.

Theorem 8.2.3 *Suppose that $\mathbb{E}|S(1)|^{2+\delta} < \infty$ for some $\delta > 0$. Then there exist $C_1, C_2 > 0$ such that for any positive sequence b_n satisfying*

$$b_n \longrightarrow \infty \quad \textit{and} \quad \log b_n = o(\log n) \qquad (n \to \infty)$$

and for any $\lambda > 0$,

$$\limsup_{n\to\infty} \frac{1}{b_n^\lambda} \log \mathbb{P}\Big\{Q_n - \mathbb{E}Q_n \le -\lambda \frac{n \log b_n}{2\pi\sqrt{\det(\Gamma)}}\Big\} \le -C_1$$

$$\liminf_{n\to\infty} \frac{1}{b_n^\lambda} \log \mathbb{P}\Big\{Q_n - \mathbb{E}Q_n \le -\lambda \frac{n \log b_n}{2\pi\sqrt{\det(\Gamma)}}\Big\} \ge -C_2.$$

This result was established in [6], Theorem 1.2. We omit its proof.

8.3 LDP of Gaussian tail in $d = 3$

According to Theorem 5.5.3, $(Q_n - \mathbb{E}Q_n)/\sqrt{n \log n}$ weakly converges to a normal distribution when $d = 3$. Naturally, a large deviation principle (also called moderate deviation principle for its connection to the weak law) is expected at least for small deviation scale b_n. Throughout this section we let $d = 3$. The following is the main result.

Theorem 8.3.1 *For any positive sequence*

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o\big(\sqrt{\log n}\big) \qquad (n \to \infty)$$

and for any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \pm (Q_n - \mathbb{E}Q_n) \ge \lambda\sqrt{n b_n \log n}\Big\} = -\lambda^2 \pi^2 \det(\Gamma). \qquad (8.3.1)$$

Proof. Our starting point is again the triangular decomposition (8.2.5). Here we take $N = [\gamma^{-1} \log_2 b_n]$ for large n, where $\log_2(\cdot)$ is the logarithmic function with the base 2, and $0 < \gamma < 1/2$ is a constant which will be specified later. Unlike the argument in the previous section, it is the first sum in the decomposition (the diagonal part) that makes major contribution.

Write

$$\overline{\eta}_j = \eta_j - \mathbb{E}\eta_j \quad \text{and} \quad \overline{\xi}_{j,k} = \xi_{j,k} - \mathbb{E}\xi_{j,k}.$$

We first prove that for any $L > 0$ there is a constant $a > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\bigg\{\bigg|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{\xi}_{j,k}\bigg| \ge a\sqrt{nb_n^3}\bigg\} \le -L. \tag{8.3.2}$$

By the fact that $2^{-\gamma} + \cdots + 2^{-\gamma N} \le 2^{-\gamma}(1-2^{-\gamma})^{-1}$ and by triangular inequality, for any $a > 0$

$$\begin{aligned} &\mathbb{P}\bigg\{\bigg|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{\xi}_{j,k}\bigg| \ge a\sqrt{nb_n^3}\bigg\} \\ &\le \sum_{j=1}^{N} \mathbb{P}\bigg\{\bigg|\sum_{k=1}^{2^{j-1}} \overline{\xi}_{j,k}\bigg| \ge a2^{\gamma}(1-2^{-\gamma})2^{-\gamma j}\sqrt{nb_n^3}\bigg\}. \end{aligned} \tag{8.3.3}$$

Let $C > 0$ be fixed but arbitrary number and set

$$\xi'_{j,k} = \overline{\xi}_{j,k} 1_{\{|\overline{\xi}_{j,k}| \le C\sqrt{n2^{-j}b_n^3}\}} - \mathbb{E}\overline{\xi}_{j,k} 1_{\{|\overline{\xi}_{j,k}| \le C\sqrt{n2^{-j}b_n^3}\}}$$

and $\xi''_{j,k} = \overline{\xi}_{j,k} - \xi'_{j,k}$. Notice that uniformly on $1 \le j \le N$,

$$\sum_{k=1}^{2^{j-1}} \mathbb{E}\xi_{j,k} = o\big(n\sqrt{2^{-j}b_n^3}\big).$$

Thus, for any $\delta > 0$,

$$\mathbb{P}\bigg\{\bigg|\sum_{k=1}^{2^{j-1}} \xi''_{j,k}\bigg| \ge \delta\sqrt{n2^{-j}b_n^3}\bigg\} \le \sum_{1\le k\le 2^{j-1}} \mathbb{P}\Big\{|\overline{\xi}_{j,k}| \ge C\sqrt{n2^{-j}b_n^3}\Big\}$$

as n is sufficiently large.

Taking $d = 3$ and $p = 2$ in (6.2.1), Theorem 6.2.1, by the in-law-identity (8.2.7) we have

$$\sup_{n,j,k} \mathbb{E}\exp\bigg\{\frac{\theta}{(n2^{-j})^{1/3}}|\overline{\xi}_{j,k}|^{2/3}\bigg\} < \infty \tag{8.3.4}$$

for some $\theta > 0$. By a standard way of using Chebyshev inequality, one can show that for any $L > 0$, there is $C > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \sup_{j,k} \mathbb{P}\Big\{|\overline{\xi}_{j,k}| \ge C\sqrt{n2^{-j}b_n^3}\Big\} \le -L.$$

Consequently, by the fact that $2^{j-1} \le 2^N \le b_n^{\gamma^{-1}}$

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \max_{1\le j\le N} \mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \xi''_{j,k}\Big| \ge \delta\sqrt{n2^{-j}b_n^3}\Big\} \le -L \tag{8.3.5}$$

for any $\delta > 0$.

By independence,

$$\mathbb{E}\exp\left\{\frac{2^{\gamma j}\theta}{\sqrt{nb_n}} \sum_{k=1}^{2^{j-1}} \xi'_{j,k}\right\} = \prod_{k=1}^{2^{j-1}} \mathbb{E}\exp\left\{\frac{2^{\gamma j}\theta}{\sqrt{nb_n}} \xi'_{j,k}\right\}.$$

For each $1 \le k \le 2^{j-1}$,

$$|\xi'_{j,k}| \le 2C\sqrt{n2^{-j}b_n^3}.$$

In view of (8.3.4), therefore, there is a $u > 0$ such that

$$\sup_{j,k,n} \mathbb{E}\exp\left\{\frac{u}{\sqrt{n2^{-j}b_n}}|\xi'_{j,k}|\right\} \le \sup_{j,k,n} \mathbb{E}\exp\left\{\frac{u(2C)^{1/3}}{(n2^{-j})^{1/3}}|\xi'_{j,k}|^{2/3}\right\} < \infty.$$

By Taylor expansion, there is a $M > 0$ such that when $|\theta|$ is sufficiently small,

$$\mathbb{E}\exp\left\{\frac{2^{\gamma j}\theta}{\sqrt{nb_n}} \xi'_{j,k}\right\} \le \exp\left\{\frac{M2^{2\gamma j}\theta^2}{nb_n} \mathbb{E}\xi_{j,k}^2\right\}.$$

By (8.2.7) $\mathbb{E}\xi_{j,k}^2 = O(n2^{-j})$. Consequently, there is a constant $M' > 0$ such that

$$\begin{aligned} &\max_{1\le j\le N} \mathbb{E}\exp\left\{\frac{\theta}{\sqrt{nb_n}} \sum_{k=1}^{2^{j-1}} \xi'_{j,k}\right\} \\ &\le \exp\left\{\frac{M'\theta^2 2^{2\gamma j}}{b_n}\right\} \le \exp\left\{\frac{M'\theta^2 2^{2\gamma N}}{b_n}\right\}. \end{aligned} \tag{8.3.6}$$

By the fact that $2^{2\gamma N} \sim b_n^2$, and by Chebyshev inequality, for any $L > 0$ there is a constant $a > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \max_{1\le j\le N} \mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \xi'_{j,k}\Big| \ge a2^{-\gamma j}\sqrt{nb_n^3}\Big\} \le -L.$$

In view of (8.3.5), therefore,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \max_{1\le j\le N} \mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \overline{\xi}_{j,k}\Big| \ge (a+\delta)2^{-\gamma j}\sqrt{nb_n^3}\Big\} \le -L. \qquad (8.3.7)$$

By (8.3.3), therefore, we have (8.3.2).

Notice that $\sqrt{nb_n^3} = o\big(\sqrt{nb_n \log n}\big)$. [2] Consequently, for any $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{\Big|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{\xi}_{j,k}\Big| \ge \epsilon\sqrt{nb_n\log n}\Big\} = -\infty.$$

To prove (8.3.1), therefore, it suffices to show that

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{\pm \sum_{j=1}^{2^N} \overline{\eta}_j \ge \lambda\sqrt{nb_n \log n}\Big\} = -\lambda^2\pi^2 \det(\Gamma). \qquad (8.3.8)$$

Write

$$\eta_j' = \overline{\eta}_j 1\Big\{|\overline{\eta}_j| \le \sqrt{2^{-N}nb_n^3 \log n}\Big\} - \mathbb{E}\overline{\eta}_j 1\Big\{|\overline{\eta}_j| \le \sqrt{2^{-N}nb_n^3\log n}\Big\},$$

$$\eta_j'' = \overline{\eta}_j 1\Big\{|\overline{\eta}_j| > \sqrt{2^{-N}nb_n^3 \log n}\Big\} - \mathbb{E}\overline{\eta}_j 1\Big\{|\overline{\eta}_j| > \sqrt{2^{-N}nb_n^3\log n}\Big\}.$$

By the in-law-identity (8.2.6) and by Theorem 6.4.1

$$\sup_{j,n} \mathbb{E}\exp\left\{\frac{\theta}{(2^{-N}n\log n)^{1/3}}|\eta_j|^{2/3}\right\} < \infty \qquad \forall\theta > 0. \qquad (8.3.9)$$

Replacing $\Big\{\overline{\xi}_{N,k};\ 1 \le k \le 2^{N-1}\Big\}$ by the sequence $\Big\{\overline{\eta}_j/\sqrt{\log n};\ 1 \le j \le 2^N\Big\}$ in the argument used for (8.3.5) gives that for any $L > 0$ there is a $a > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{\Big|\sum_{j=1}^{2^N} \eta_j''\Big| \ge a\sqrt{2^{-N}nb_n^3\log n}\Big\} \le -L$$

We now make $\gamma < 1/4$. Then

$$\sqrt{2^{-N}nb_n^3 \log n} = o\left(\sqrt{\frac{n\log n}{b_n}}\right) \qquad (n\to\infty). \qquad (8.3.10)$$

[2] This is the only place in the whole proof where the assumption $b_n = o(\sqrt{\log n})$ is needed.

In particular,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\Big|\sum_{j=1}^{2^N}\eta_j''\Big|\ge\epsilon\sqrt{nb_n\log n}\Big\}=-\infty.$$

Therefore, it remains to show that

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\pm\sum_{j=1}^{2^N}\eta_j'\ge\lambda\sqrt{nb_n\log n}\Big\}=-\lambda^2\pi^2\det(\Gamma). \tag{8.3.11}$$

By Theorem 1.1.4, all we need is to establish

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n\log n}}\sum_{j=1}^{2^N}\eta_j'\right\}=\frac{\theta^2}{(2\pi)^2\det(\Gamma)} \tag{8.3.12}$$

for every real number θ.

By independence, Taylor expansion and (8.3.10),

$$\begin{aligned}\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n\log n}}\sum_{j=1}^{2^N}\eta_j'\right\}&=\prod_{j=1}^{2^N}\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n\log n}}\eta_j'\right\}\\&=\prod_{j=1}^{2^N}\exp\left\{\big(1+o(1)\big)\frac{\theta^2b_n}{2n\log n}\mathbb{E}(\eta_j')^2\right\}\qquad(n\to\infty).\end{aligned}$$

Observe that from (8.2.6) that uniformly on $1\le j\le 2^N$,

$$\begin{aligned}\mathbb{E}|\overline{\eta}_j|&\sim\mathbb{E}|Q_{[2^{-N}n]}-\mathbb{E}Q_{[2^{-N}n]}|\\&=O\big(\sqrt{2^{-N}n\log n}\big)=o\big(\sqrt{2^{-N}nb_n^3\log n}\big).\end{aligned}$$

Consequently, by the weak law given in Theorem 5.5.3

$$\mathbb{E}(\eta_j')^2\sim\mathbb{E}(\overline{\eta}_j)^2\sim\frac{1}{(2\pi)^2\det(\Gamma)}2^{-N}n\log n$$

uniformly over $1\le j\le 2^N$ as $n\to\infty$. This leads to (8.3.12). □

If we only consider the lower tail of $Q_n-\mathbb{E}Q_n$ in exchange for relaxing the condition on b_n, we establish the following sub-Gaussian tail.

Theorem 8.3.2 *For any positive sequence*

$$b_n\to\infty\quad\textit{and}\quad b_n=o(n)\qquad(n\to\infty)$$

and for every $\lambda > 0$,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{Q_n - \mathbb{E}Q_n \le -\lambda\sqrt{nb_n \log(n/b_n)}\Big\} \tag{8.3.13}$$
$$\le -\lambda^2\pi^2 \det(\Gamma).$$

Proof. The main idea in the remaining argument is sub-additivity. Introduce the sequence,

$$q(n) = \sum_{j=1}^{n}\sum_{k=j}^{\infty} \mathbb{P}\{S(k) = 0\} \qquad n = 1, 2, \cdots .$$

By an argument used in the proof of (5.1.6),

$$q(n) \sim \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Gamma)}} \sum_{j=1}^{n}\sum_{k=j}^{\infty} \frac{1}{k^{3/2}} \sim \frac{4}{(2\pi)^{3/2}\sqrt{\det(\Gamma)}}\sqrt{n}. \tag{8.3.14}$$

We claim that for any integers $m, n \ge 1$,

$$\mathbb{E}\bigg(\sum_{j=1}^{n}\sum_{k=n+1}^{m+n} 1_{\{S(j)=S(k)\}}\bigg) = q(n) + q(m) - q(m+n). \tag{8.3.15}$$

Indeed,

$$\begin{aligned}
&\mathbb{E}\bigg(\sum_{j=1}^{n}\sum_{k=n+1}^{m+n} 1_{\{S(j)=S(k)\}}\bigg)\\
&= \sum_{j=1}^{n}\sum_{k=0}^{m-1} \mathbb{P}\{S_{j+k} = 0\} = \sum_{j=1}^{n}\sum_{k=j}^{m+j-1} \mathbb{P}\{S_k = 0\}\\
&= \sum_{j=1}^{n}\bigg(\sum_{k=j}^{\infty} \mathbb{P}\{S_k = 0\} - \sum_{k=m+j}^{\infty} \mathbb{P}\{S_k = 0\}\bigg)\\
&= \sum_{j=1}^{n}\sum_{k=j}^{\infty} \mathbb{P}\{S_k = 0\} - \sum_{j=m+1}^{m+n}\sum_{k=j}^{\infty} \mathbb{P}\{S_k = 0\}\\
&= q(n) + q(m) - q(m+n).
\end{aligned}$$

Write

$$\xi(n) = q(n) + (\mathbb{E}Q_n - Q_n) \qquad n = 1, 2, \cdots .$$

Then

$$\begin{aligned}\xi(m+n) &= q(m+n) + (\mathbb{E}Q_{m+n} - Q_{m+n})\\ &= q(n) + q(m) - \mathbb{E}\Big(\sum_{j=1}^{n}\sum_{k=n+1}^{m+n} 1_{\{S_j=S_k\}}\Big)\\ &+ (\mathbb{E}Q_n - Q_n) + (\mathbb{E}\widetilde{Q}_m - \widetilde{Q}_m)\\ &- \sum_{j=1}^{n}\sum_{k=n+1}^{m+n} 1_{\{S_j=S_k\}} + \mathbb{E}\Big(\sum_{j=1}^{n}\sum_{k=n+1}^{m+n} 1_{\{S_j=S_k\}}\Big)\\ &\le \Big\{q(n) + (\mathbb{E}Q_n - Q_n)\Big\} + \Big\{q(m) + (\mathbb{E}\widetilde{Q}_m - \widetilde{Q}_m)\Big\}\end{aligned}$$

where

$$\widetilde{Q}_m = \sum_{n+1\le j<k\le n+m} 1_{\{S(j)=S(k)\}}$$

is independent of $\{S_1, \cdots, S_n\}$ and is equal in law to Q_m.

Therefore, $\xi(n)$ is a sub-additive sequence:

$$\xi(n+m) \le \xi(n) + \widetilde{\xi}(m) \qquad m, n \ge 1 \tag{8.3.16}$$

with $\widetilde{\xi}(m) \overset{d}{=} \xi(m)$ and with independence between $\widetilde{\xi}(m)$ and $\{\xi(1), \cdots, \xi(n)\}$.

By straight forward calculation we have that for each $k \ge 1$,

$$\xi(k) - \xi(k-1) = \sum_{j=1}^{\infty} \mathbb{P}\{S(j) = 0\} - \sum_{j=1}^{k-1} 1_{\{S(j)=S(k)\}}.$$

Consequently,

$$\sup_{k\ge 1}\big\{\xi(k) - \xi(k-1)\big\} \le \sum_{k=1}^{\infty} \mathbb{P}\{S(k) = 0\}.$$

By transience in $d = 3$, the right hand side is a finite constant. By Theorem 6.4.3 and (8.3.14), the normalized sequence

$$\max_{1\le k\le n} \xi(k)/\sqrt{n\log n} \qquad n = 2, 3, \cdots$$

is bounded in probability.

By Theorem 1.3.3,

$$\sup_{n} \mathbb{E}\exp\Big\{\frac{\theta}{\sqrt{n\log n}}\xi(n)\Big\} < \infty \qquad \forall\theta > 0. \tag{8.3.17}$$

Let $\theta > 0$. By sub-additivity given in (8.3.16),

$$\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n\log(n/b_n)}}\xi(n)\right\} \le \left(\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n\log(n/b_n)}}\xi([n/b_n])\right\}\right)^{1+b_n}.$$

By the central limit theorem given in Theorem 5.5.3, by (8.3.17) (with n being replaced by $[n/b_n]$) and by dominated convergence theorem,

$$\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n\log(n/b_n)}}\xi([n/b_n])\right\} \longrightarrow \exp\left\{\frac{\theta^2}{(2\pi)^2\det(\Gamma)}\right\} \qquad (n\to\infty).$$

Consequently, for any $\theta > 0$

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n\log(n/b_n)}}\xi(n)\right\} \le \frac{\theta^2}{(2\pi)^2\det(\Gamma)}.$$

In view of (8.3.14), this is equivalent to

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\sqrt{\frac{b_n}{n\log(n/b_n)}}(\mathbb{E}Q_n - Q_n)\right\} \le \frac{\theta^2}{(2\pi)^2\det(\Gamma)}$$

which leads to the conclusion in Theorem 8.3.2 by a standard application of Chebyshev inequality. □

We conjecture that (8.3.13) can be strengthened into

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\left\{Q_n - \mathbb{E}Q_n \le -\lambda\sqrt{nb_n\log(n/b_n)}\right\} = -\lambda^2\pi^2\det(\Gamma).$$

8.4 LDP of non-Gaussian tail in $d = 3$

The decomposition (8.2.5) shows that the self-intersection of a random walk consists of two parts: short and long range intersections. When $b_n = o(\sqrt{\log n})$, it is the short range intersection that dominates. Consequently, the large deviation principle for $Q_n - \mathbb{E}Q_n$ has a Gaussian tail in this case. In this section, we shall see that increasing the scale b_n may change the dynamics.

Throughout this section, $d = 3$.

Lemma 8.4.1 *For any $L > 0$, there is a $\lambda > 0$ such that*

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\left\{|Q_n - \mathbb{E}Q_n| \ge \lambda\sqrt{nb_n^3}\right\} \le -L$$

for any $b_n = o(n)$ with $\lim_{n\to\infty} b_n/\sqrt{\log n} = \infty$.

Proof. Again, our starting point is the decomposition (8.2.5). Here we let $N = [\log_2(\log n)]$ (Recall that $\log_2(\cdot)$ is the logarithmic function with base 2). One can see that (8.3.2) remains true. Here we emphasize two facts: First, the constant $\lambda > 0$ exists uniformly for b_n satisfying the condition posted in the lemma.[3] Second, to establish (8.3.2) in current setting, we take $\gamma = 1/2$ in (8.3.3), and a place we use our assumption $\lim_{n\to\infty} b_n/\sqrt{\log n} = \infty$ is in (8.3.6), where the right hand side (note $\gamma = 1/2$) is bounded by

$$\exp\left\{\frac{M'\theta^2 \log n}{b_n}\right\} = \exp\{o(b_n)\} \qquad (n \to \infty).$$

It remains to show that for any $L > 0$, there is a $\lambda > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left\{\left|\sum_{j=1}^{2^N} \overline{\eta}_j\right| \geq \lambda\sqrt{nb_n^3}\right\} \leq -L. \tag{8.4.1}$$

Write

$$\zeta_j = \eta_j/\sqrt{\log n} \qquad j = 1, \cdots, 2^N.$$

From (8.3.9),

$$\sup_{n,j} \mathbb{E}\exp\left\{\frac{\theta}{(2^{-N}n)^{1/3}}|\zeta_j|^{2/3}\right\} < \infty.$$

The argument used for (8.3.5) (Once again, $\gamma = 1/2$, for which the assumption on b_n is needed here for similar reason) leads to that for any $L > 0$, there is a $\lambda > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left\{\left|\sum_{j=1}^{2^N} \eta_j\right| \geq \lambda\sqrt{2^{-N}nb_n^3}\right\} \leq -L.$$

Equivalently, we have proved (8.4.1). □

The following is the main result of this section.

Theorem 8.4.2 *For any b_n satisfying*

$$\lim_{n\to\infty} b_n/\sqrt{\log n} = \infty \quad \textit{and} \quad b_n = o(n) \qquad (n \to \infty)$$

[3]When the assumption on b_n is weakened as $\liminf_{n\to\infty} b_n/\sqrt{\log n} > 0$, (8.3.2) remains true, but we are unable to establish its uniformity over b_n.

and for any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{Q_n - \mathbb{E}Q_n \geq \lambda\sqrt{nb_n^3}\Big\} \tag{8.4.2}$$
$$= -\sqrt[3]{\frac{\det(\Gamma)}{2}}\kappa(3,2)^{-8/3}\lambda^{2/3}.$$

Here we recall that under the definition given in Remark 3.3.3, $\kappa(3,2) > 0$ *is the best constant of Gagliardo-Nirenberg inequality in* $\mathbb{R}^3$*:*

$$||f||_4 \leq C||\nabla f||_2^{3/4}||f||_2^{1/4} \qquad f \in W^{1,2}(\mathbb{R}^3).$$

Proof. Under our assumption on b_n, $\sqrt{nb_n\log(n/b_n)} = o\big(\sqrt{nb_n^3}\big)$ as $n \to \infty$. By the sub-Gaussian tail given in Theorem 8.3.2,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{Q_n - \mathbb{E}Q_n \leq -\lambda\sqrt{nb_n^3}\Big\} = -\infty \qquad \forall\lambda > 0.$$

Consequently, (8.4.2) is equivalent to

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{|Q_n - \mathbb{E}Q_n| \geq \lambda\sqrt{nb_n^3}\Big\} \tag{8.4.3}$$
$$= -\sqrt[3]{\frac{\det(\Gamma)}{2}}\kappa(3,2)^{-8/3}\lambda^{2/3}.$$

We now claim that the statement in Lemma 8.4.1 can be slightly strengthened into: There is a constant $C_Q > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{|Q_n - \mathbb{E}Q_n| \geq \lambda\sqrt{nb_n^3}\Big\} \leq -C_Q\lambda^{2/3} \qquad \forall\lambda > 0 \tag{8.4.4}$$

for any b_n with $b_n/\sqrt{\log n} \to \infty$.

Indeed, by Lemma 8.4.1 there is a constant $a > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{|Q_n - \mathbb{E}Q_n| \geq a\sqrt{nb_n^3}\Big\} \leq -1$$

for every $b_n = o(n)$ with $b_n/\sqrt{\log n} \to \infty$. Replacing b_n by $(a^{-1}\lambda)^{2/3}b_n$ in above bound and letting $C_Q = a^{-2/3}$ we have (8.4.4).

By Theorem 1.1.4, a possible approach is to compute the logarithmic generating function

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}|Q_n - \mathbb{E}Q_n|^{1/2}\right\}.$$

Unfortunately, we are not able to do this. As a matter of the fact, we even don't know whether or not the limsup

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp \left\{ \theta \Big(\frac{b_n}{n}\Big)^{1/4} |Q_n - \mathbb{E}Q_n|^{1/2} \right\}$$

is finite for any $\theta > 0$, despite of (8.4.4).

Instead, we condition on that $|Q_n - \mathbb{E}Q_n|$ is suitably bounded. More precisely, let $M > 0$ be large but fixed and define the event

$$\Omega_n = \Big\{ |Q_n - \mathbb{E}Q_n| \le M\sqrt{nb_n^3} \Big\}.$$

We prove that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\left(\exp \left\{ \theta \Big(\frac{b_n}{n}\Big)^{1/4} |Q_n - \mathbb{E}Q_n|^{1/2} \right\} \Big| \Omega_n \right) \\ &= \frac{1}{16} \Big(\frac{3}{2}\Big)^3 \det(\Gamma)^{-1} \kappa(3,2)^8 \theta^4 \qquad 0 < \theta < cM^{1/6} \end{aligned} \tag{8.4.5}$$

where $c > 0$ is a constant independent of M.

The triangular decomposition (8.2.5) plays an important role in our proof of (8.4.5). As in section 8.2, we let N be large but fixed. By (8.2.6) and (8.4.4),

$$\limsup_{n\to\infty} \log \mathbb{P}\Big\{ |\overline{\eta}_j| \ge \lambda\sqrt{nb_n^3} \Big\} \le -C_Q 2^{N/3} \lambda^{2/3}$$

for every $\lambda > 0$ (recall that $\overline{\eta}_j = \eta_j - \mathbb{E}\eta_j$). By Theorem 1.2.2,

$$\limsup_{n\to\infty} \log \mathbb{P}\Big\{ \Big| \sum_{j=1}^{2^N} \overline{\eta}_j \Big| \ge \lambda\sqrt{nb_n^3} \Big\} \le -C_Q 2^{N/3} \lambda^{2/3}$$

for every $\lambda > 0$. In view of (8.4.4),

$$\limsup_{n\to\infty} \log \mathbb{P}\Big\{ \Big| \sum_{j=1}^{2^N} \overline{\eta}_j \Big| \ge \lambda\sqrt{nb_n^3} \Big| \Omega_n \Big\} \le -C_Q 2^{N/3} \lambda^{2/3}. \tag{8.4.6}$$

We now claim that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\left(\exp \left\{ \theta \Big(\frac{b_n}{n}\Big)^{1/4} \Big| \sum_{j=1}^{2^N} \overline{\eta}_j \Big|^{1/2} \right\} \Big| \Omega_n \right) \le \frac{27}{256} 2^{-N} C_Q^{-3} \theta^4. \tag{8.4.7}$$

By (8.2.7) and by (6.2.1) in Theorem6.2.1 , there is a $\theta > 0$ (possibly depends on N) such that

$$\sup_{j,k,n} \mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n}}\xi_{j,k}^{2/3}\right\} < \infty$$

for some $\theta > 0$. Notice that N is fixed. By decomposition (8.2.5)

$$\sup_n \mathbb{E}\exp\left\{\frac{\theta}{\sqrt[3]{n}}\Big||Q_n - \mathbb{E}Q_n| - \Big|\sum_{j=1}^{2^N}\overline{\eta}_j\Big|\Big|^{2/3}\right\} < \infty$$

for some $\theta > 0$. This leads to, by a Chebyshev type argument, that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big||Q_n - \mathbb{E}Q_n| - \Big|\sum_{j=1}^{2^N}\overline{\eta}_j\Big|\Big|^{1/2}\right\} < \infty$$

for every $\theta > 0$. Consequently,

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left(\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big|\sum_{j=1}^{2^N}\overline{\eta}_j\Big|^{1/2}\right\}\Big|\Omega_n\right) < \infty \qquad \forall\theta > 0.$$

Combined with the observation (8.4.6), this legitimizes the use of Varadhan's integral lemma (Theorem 1.1.6). Consequently, for any $\theta > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left(\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big|\sum_{j=1}^{2^N}\overline{\eta}_j\Big|^{1/2}\right\}\Big|\Omega_n\right)\\
&\le \sup_{\lambda>0}\left\{\lambda\theta - C2^{N/3}\lambda^{4/3}\right\} = \frac{27}{256}2^{-N}C_Q^{-3}\theta^4.
\end{aligned}$$

We have confirmed (8.4.7).

Recall (Theorem 8.1.2, with $d = 3$ and $p = 2$) that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\} \qquad (8.4.8)\\
&= \sup_{g\in\mathcal{F}_3}\left\{\theta\det(\Gamma)^{-1/4}\Big(\int_{\mathbb{R}^3}|g(x,\epsilon)|^4dx\Big)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla g(x)|^2dx\right\}\\
&= M_\epsilon(\theta) \qquad \text{(say)}
\end{aligned}$$

where

$$l(n,x,\epsilon) = \det(\Gamma)^{-1/2}\Big(\frac{b_n}{n}\Big)^{3/2}\sum_{k=1}^n p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}\big(S(k)-x\big)\Big)$$

$$\mathcal{F}_3 = \Big\{g \in \mathcal{L}^2(\mathbb{R}^3);\quad ||g||_2 = 1 \text{ and } \nabla g \in \mathcal{L}^2(\mathbb{R}^3)\Big\}$$

and for $g \in \mathcal{F}_3$,

$$g(x,\epsilon) = \left\{\int_{\mathbb{R}^3} p_\epsilon(x-y)g^2(y)dy\right\}^{1/2}.$$

Clearly, (8.4.8) implies that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}\Big|\Omega_n\right] \tag{8.4.9}$$
$$\le M_\epsilon(\theta).$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}\\
&= \mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}1_{\Omega_n}\right]\\
&+\mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}1_{\Omega_n^c}\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\max\left\{\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}\Big|\Omega_n\right],\right.\\
&\left.\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}1_{\Omega_n^c}\right]\right\}\\
&\ge M_\epsilon(\theta).
\end{aligned}$$

By Cauchy-Schwartz inequality,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}1_{\Omega_n^c}\right]\\
&\le\left\{\mathbb{E}\exp\left\{2\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}\right]^{1/2}\Big\{\mathbb{P}(\Omega_n^c)\Big\}^{1/2}.
\end{aligned}$$

Combining this with (8.4.8) (with θ be replaced by 2θ) and (8.4.4) (with $\lambda = M$),

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}1_{\Omega_n^c}\right]$$
$$\le \frac{1}{2}M_\epsilon(2\theta) - \frac{1}{2}C_Q M^{2/3}.$$

By Jensen inequality,

$$M_\epsilon(2\theta) \le \sup_{g\in\mathcal{F}_3}\left\{2\theta\det(\Gamma)^{-1/4}\Big(\int_{\mathbb{R}^3}|g(x)|^4dx\Big)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla g(x)|^2dx\right\}.$$

According to Theorem C.1 in Appendix, for any $\theta > 0$

$$\sup_{g\in\mathcal{F}_3}\left\{\theta\Big(\int_{\mathbb{R}^3}|g(x)|^4dx\Big)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla g(x)|^2dx\right\} \tag{8.4.10}$$
$$= \frac{1}{4}\Big(\frac{3}{2}\Big)^3\kappa(3,2)^8\theta^4.$$

Hence, we have proved that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} \xi^2(n,x,\epsilon)\Big)^{1/2}\right\}1_{\Omega_n^c}\right]$$
$$\le \frac{27}{4}\det(\Gamma)^{-1}\kappa(3,2)^8\theta^4 - \frac{1}{2}C_Q M^{2/3}.$$

Notice that there exists a constant $c > 0$ independent of M, such that the right hand side is negative for all $0 < \theta < cM^{1/6}$. Summarizing our argument after (8.4.9), therefore,

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}\Big|\Omega_n\right] \tag{8.4.11}$$
$$\ge M_\epsilon(\theta)$$

for $0 < \theta < cM^{1/6}$.

Combining (8.4.9) and (8.4.11), for any $0 < \theta < cM^{1/6}$

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{x\in\mathbb{Z}^3} l^2(n,x,\epsilon)\Big)^{1/2}\right\}\Big|\Omega_n\right] \tag{8.4.12}$$
$$= M_\epsilon(\theta).$$

As in the proof of Theorem 8.2.1, a key idea is the exponential approximation. For this we need the decomposition given in (8.2.9). Similar to (8.2.14), for any $\theta > 0$

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{j=1}^{2^N}\eta_j^2(\epsilon)\Big)^{1/2}\right\} \tag{8.4.13}$$
$$\le \frac{1}{2^{N+2}}\Big(\frac{3}{2}\Big)^3\det(\Gamma)^{-1}\kappa(3,2)^8\theta^4.$$

Similar to (8.2.15),

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\Big(\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}}\big|\xi_{j,k}-\xi_{j,k}(\epsilon)\big|\Big)^{1/2}\right\}=0. \tag{8.4.14}$$

In addition, by (8.2.7) one can see that

$$\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}}\mathbb{E}\xi_{j,k}=O(\sqrt{n})\qquad (n\to\infty).$$

Combining this with (8.4.7), (8.4.13) and (8.4.14), applying Hölder inequality we have that for any $\theta > 0$

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\Bigg[\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}\right.$$
$$\left.\times\Big|(Q_n-\mathbb{E}Q_n)-\frac{1}{2}\sum_{x\in\mathbb{Z}^3}\xi^2(n,x,\epsilon)\Big|^{1/2}\right\}\Bigg|\Omega_n\Bigg]=0.$$

In view of (8.4.12) (with θ be replaced by $\theta/\sqrt{2}$), the exponential approximation established in above gives that for $0<\theta<cM^{1/6}$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left(\exp\left\{\theta\Big(\frac{b_n}{n}\Big)^{1/4}|Q_n-\mathbb{E}Q_n|^{1/2}\right\}\Big|\Omega_n\right)$$
$$=\sup_{g\in\mathcal{F}_3}\left\{\frac{\theta}{\sqrt{2}}\det(\Gamma)^{-1/4}\Big(\int_{\mathbb{R}^3}|g(x)|^4dx\Big)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla g(x)|^2dx\right\}$$
$$=\frac{1}{16}\Big(\frac{3}{2}\Big)^3\det(\Gamma)^{-1}\kappa(3,2)^8\theta^4$$

where the last step follows from (8.4.10).

We have completed the proof of (8.4.5).

We now finish the proof of Theorem 8.4.2 or equivalently, (8.4.3). Since (8.4.5) does not hold for all $\theta > 0$, Gärtner-Ellis theorem does not directly apply to our setting. However, by Chebyshev inequality we have the following upper bound

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{|Q_n-\mathbb{E}Q_n|\ge\lambda\sqrt{nb_n^3}\Big|\Omega_n\Big\}\\
&\le-\sup_{0<\theta<cM^{1/6}}\Big\{\theta\lambda^{1/2}-\frac{1}{16}\Big(\frac{3}{2}\Big)^3\det(\Gamma)^{-1}\kappa(3,2)^8\theta^4\Big\}\\
&=-\sqrt[3]{\frac{\det(\Gamma)}{2}}\kappa(3,2)^{-8/3}\lambda^{2/3}
\end{aligned}$$

for

$$0<\lambda<\frac{1}{16}\Big(\frac{3}{2}\Big)^6\det(\Gamma)^{-2}\kappa(3,2)^{16}c^6M.$$

Since $M > 0$ can be arbitrarily large, by (8.4.4) (with $\lambda = M$) and a standard argument of exponential approximation,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{|Q_n-\mathbb{E}Q_n|\ge\lambda\sqrt{nb_n^3}\Big\}\\
&\le-\sqrt[3]{\frac{\det(\Gamma)}{2}}\kappa(3,2)^{-8/3}\lambda^{2/3}\qquad\forall\lambda>0.
\end{aligned}\tag{8.4.15}$$

To establish the lower bound, we introduce an i.i.d. sequence $\{\tau_k\}$ of exponential random variables with parameter 1. We assume independence between $\{\tau_k\}$ and $\{S(k)\}$. Write

$$Y_n=\frac{1}{b_n}\sum_{k=1}^{[b_n]}\tau_k\qquad n=1,2,\cdots.$$

By (8.4.5),

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\Bigg[\exp\Big\{\theta b_n\Big(\frac{|Q_n-\mathbb{E}Q_n|^{1/2}}{\sqrt[4]{nb_n^3}}+\frac{Y_n}{cM^{1/6}}\Big)\Big\}\Bigg|\Omega_n\Bigg]=\Lambda(\theta)$$

for every $\theta > 0$, where

$$\Lambda(\theta)=\begin{cases}\frac{1}{16}\Big(\frac{3}{2}\Big)^3\det(\Gamma)^{-1}\kappa(3,2)^8\theta^4+\Big(1-\frac{\theta}{cM^{1/6}}\Big)^{-1} & 0<\theta<cM^{1/6}\\[2ex] \infty & \theta\ge cM^{1/6}.\end{cases}$$

According to Definition 1.1.3, one can see that the logarithmic moment generating function $\Lambda(\theta)$ is essentially smooth.

Applying Theorem 1.2.3 to the random sequence

$$Z_n = \frac{|Q_n - \mathbb{E}Q_n|^{1/2}}{\sqrt[4]{nb_n^3}} + \frac{Y_n}{cM^{1/6}} \qquad n = 1, 2, \cdots$$

gives that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\bigg\{ \frac{|Q_n - \mathbb{E}Q_n|^{1/2}}{\sqrt[4]{nb_n^3}} + \frac{Y_n}{cM^{1/6}} \ge \lambda \bigg| \Omega_n \bigg\} \\
&= \lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{ Z_n \ge \lambda \big| \Omega_n \big\} = -I(\lambda) \qquad \forall \lambda > 0
\end{aligned}$$

where for every $\lambda > 0$,

$$\begin{aligned}
I(\lambda) &= \sup_{\theta\in\mathbb{R}} \Big\{ \theta\lambda - \Lambda(|\theta|) \Big\} = \sup_{\theta>0} \Big\{ \theta\lambda - \Lambda(\theta) \Big\} \\
&= \sup_{0<\theta<cM^{1/6}} \bigg\{ \theta\lambda - \frac{1}{16}\Big(\frac{3}{2}\Big)^3 \det(\Gamma)^{-1}\kappa(3,2)^8\theta^4 - \Big(1 - \frac{\theta}{cM^{1/6}}\Big)^{-1} \bigg\} \\
&\le \sup_{0<\theta<cM^{1/6}} \bigg\{ \theta\lambda - \frac{1}{16}\Big(\frac{3}{2}\Big)^3 \det(\Gamma)^{-1}\kappa(3,2)^8\theta^4 \bigg\}.
\end{aligned}$$

It is straightforward to check that the right hand side is equal to

$$\sqrt[3]{\frac{\det(\Gamma)}{2}}\kappa(3,2)^{-8/3}\lambda^{4/3}$$

when

$$0 < \lambda < \frac{1}{4}\Big(\frac{3}{2}\Big)^3 \det(\Gamma)^{-1}\kappa(3,2)^8 c^3 M^{1/2}.$$

For such λ we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\bigg\{ \frac{|Q_n - \mathbb{E}Q_n|^{1/2}}{\sqrt[4]{nb_n^3}} + \frac{Y_n}{cM^{1/6}} \ge \lambda \bigg| \Omega_n \bigg\} \\
&\ge -\sqrt[3]{\frac{\det(\Gamma)}{2}}\kappa(3,2)^{-8/3}\lambda^{4/3}.
\end{aligned} \tag{8.4.16}$$

In addition, for any $\epsilon > 0$ by Cramér's large deviation principle (Theorem 1.1.5), one concludes that

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ Y_n \ge \epsilon c M^{1/6} \Big\} = -\infty.$$

Combining this with (8.4.4) (with $\lambda = M$) and (8.4.16), a standard argument by exponential approximation gives

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ |Q_n - \mathbb{E}Q_n|^{1/2} \ge \lambda \sqrt[4]{nb_n^3} \Big\} \ge -\sqrt[3]{\frac{\det(\Gamma)}{2}}\kappa(3,2)^{-8/3}\lambda^{4/3}.$$

Replacing λ by $\sqrt{\lambda}$,

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{|Q_n - \mathbb{E}Q_n| \ge \lambda\sqrt{nb_n^3}\Big\} \tag{8.4.17}$$
$$\ge -\sqrt[3]{\frac{\det(\Gamma)}{2}}\kappa(3,2)^{-8/3}\lambda^{2/3} \qquad \forall\lambda > 0.$$

Finally, the desired (8.4.3) follows from (8.4.15) and (8.4.17). $\square$

8.5 LDP for renormalized range in $d = 2, 3$

Recall that by our notation, the range of a lattice valued random walk $\{S(k)\}$ is given by

$$R_n = \#\big\{S[1,n]\big\} = \#\big\{S(1), \cdots, S(n)\big\}, \qquad n = 1, 2, \cdots.$$

In this section we study the large deviations for the renormalized range $R_n - \mathbb{E}R_n$.

It is intuitively clear that Q_n and R_n are negatively correlated. Indeed, a spread-out trajectory rarely intersects. As a consequence, we shall see that, under the large deviations of asymmetric tails, the behavior $R_n - \mathbb{E}R_n$ is naturally link to that of $-(Q_n - \mathbb{E}Q_n)$.

Similar to (8.2.5), given an integer $N \ge 1$,

$$R_n = \sum_{j=1}^{2^N} \beta_j - \sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \alpha_{j,k} \tag{8.5.1}$$

where

$$\beta_j = \#\Big\{S\big((j-1)n2^{-N}, jn2^{-N}\big]\Big\}$$

$$\alpha_{j,k} = \#\Big\{S\big((2k-2)n2^{-j}, (2k-1)n2^{-j}\big] \cap S\big((2k-1)n2^{-j}, (2k)n2^{-j}\big]\Big\}.$$

The sequence $\{\beta_j\}_{1\le j\le 2^N}$ is an independent sequence with

$$\beta_j \overset{d}{=} R_{n_j} \qquad j = 1, \cdots, 2^N \tag{8.5.2}$$

where

$$n_j = \#\big((j-1)n2^{-N}, jn2^{-N}\big], \qquad j = 1, \cdots, 2^N.$$

For each $1 \le j \le N$, $\{\alpha_{j,k}\}_{1\le k\le 2^{j-1}}$ is an independent sequence with

$$\alpha_{j,k} \overset{d}{=} \#\Big\{S\big[0, n_{j,k}-1\big] \cap \widetilde{S}\big[1, n'_{j,k}\big]\Big\} \qquad k = 1, \cdots 2^{j-1} \tag{8.5.3}$$

where $\{\widetilde{S}(k)\}$ is an independent copy of $\{S(k)\}$ and

$$n_{j,k} = \#\big\{(2k-1)n2^{-j}, (2k)n2^{-j}\big]$$

$$n'_{j,k} = \#\big\{(2k)n2^{-j}, (2k+1)n2^{-j}\big].$$

Corespondent to Theorem 8.2.3, we have

Theorem 8.5.1 *Let $d=2$. Suppose that Suppose that $\mathbb{E}|S(1)|^{2+\delta} < \infty$ for some $\delta > 0$. Then there exist $K_1, K_2 > 0$ such that for any positive sequence b_n satisfying*

$$b_n \longrightarrow \infty \quad \textit{and} \quad \log b_n = o(\sqrt{\log n}) \qquad (n \to \infty)$$

and for any $\lambda > 0$,

$$\limsup_{n\to\infty} \frac{1}{b_n^\lambda} \log \mathbb{P}\Big\{R_n - \mathbb{E}R_n \ge 2\lambda\pi\sqrt{\det(\Gamma)}\frac{n\log b_n}{(\log n)^2}\Big\} \le -K_1,$$

$$\liminf_{n\to\infty} \frac{1}{b_n^\lambda} \log \mathbb{P}\Big\{R_n - \mathbb{E}R_n \ge 2\lambda\pi\sqrt{\det(\Gamma)}\frac{n\log b_n}{(logn)^2}\Big\} \ge -K_2.$$

The proof is omitted here and the reader is referred to [7] (Corollary 1.3) for a proof.

The following theorem is an analogue to the results given in Theorem 8.3.1 and Theorem 8.3.2.

Theorem 8.5.2 *Let $d=3$.*

(1). For any positive sequence b_n satisfying

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o\big(\sqrt{\log n}\big) \qquad (n \to \infty)$$

and for any $\lambda > 0$,

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \pm (R_n - \mathbb{E}R_n) \ge \lambda\sqrt{nb_n \log n}\Big\} \\ &= -\gamma_S^{-2}\lambda^2\pi^2 \det(\Gamma).\end{aligned} \tag{8.5.4}$$

(2). If the condition on b_n is relaxed to

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o\big(n\big) \qquad (n \to \infty)$$

we have

$$. \limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ R_n - \mathbb{E}R_n \geq \lambda\sqrt{nb_n \log(n/b_n)}\Big\} \\ \leq -\gamma_S^{-2}\lambda^2\pi^2 \det(\Gamma) \tag{8.5.5}$$

for every $\lambda > 0$. *Here we recall our notation*

$$\gamma_S = \mathbb{P}\{S(k) \neq 0; \quad \forall k \geq 1\}.$$

Proof. Replacing the decomposition (8.2.5) by (8.5.1), the proof of (8.5.4) is an obvious modification of the argument used in the proof of Theorem 8.3.1. We now prove (8.5.5). Recall that

$$q(n) = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \mathbb{P}\{S(k) = 0\}.$$

First notice that

$$\sum_{j=1}^{n}\sum_{k=n+1}^{m+n} 1_{\{S_j = S_k\}} = \sum_x \sum_{j=1}^{n}\sum_{k=n+1}^{m+n} 1_{\{S_j=S_k=x\}} \\ \geq \sum_x 1_{x\in S[1,n]} 1_{\{x\in S[n+1,n+m]\}} = \#\big\{S[1,n]\cap S[n+1,n+m]\big\}.$$

By (8.3.14), for any integers $m, n \geq 1$,

$$\mathbb{E}\#\big\{S[1,n]\cap S[n+1,n+m]\big\} \\ \leq \mathbb{E}\sum_{j=1}^{n}\sum_{k=n+1}^{m+n} 1_{\{S_j=S_k\}} = q(n) + q(m) - q(m+n).$$

Notice that

$$R_{m+n} = R_n + \#\big\{S[n+1,n+m]\big\} - \#\big\{S[1,n]\cap S[n+1,n+m]\big\}.$$

Similar to (8.3.16), therefore, the sequence $\{\eta(n)\}$ defined by

$$\eta(n) = q(n) + R_n - \mathbb{E}R_n \qquad n = 1, 2, \cdots$$

is sub-additive:

$$\eta(m+n) \leq \eta(n) + \widetilde{\eta}(m) \qquad m, n \geq 1 \tag{8.5.6}$$

where

$$\widetilde{\eta}(m) = q(m) + \#\big\{S[n+1,n+m]\big\} - \mathbb{E}\#\big\{S[n+1,n+m]\big\}$$

is independent of $\{\eta(1),\cdots,\eta(n)\}$ and has the same distribution as $\eta(m)$.

The rest of the proof can be carried out in the same way as the proof of Theorem 8.3.2. □

Perhaps the following is the the most interesting result of this section.

Theorem 8.5.3 *Let $\lambda>0$ be fixed but arbitrary.*

When $d=2$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{R_n-\mathbb{E}R_n\le-\lambda\frac{nb_n}{(\log n)^2}\Big\} \tag{8.5.7}$$
$$=-(2\pi)^2\det(\Gamma)^{-1/2}\kappa(2,2)^{-4}\lambda$$

for any positive sequence b_n satisfying

$$b_n\longrightarrow\infty \quad and \quad b_n=o\big(\log n\big) \qquad (n\to\infty).$$

When $d=3$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{R_n-\mathbb{E}R_n\le-\lambda\sqrt{nb_n^3}\Big\} \tag{8.5.8}$$
$$=-\sqrt[3]{\frac{\det(\Gamma)}{2}}\kappa(3,2)^{-8/3}\gamma_S^{-4/3}\lambda^{2/3}$$

for any positive sequence b_n satisfying

$$\lim_{n\to\infty}b_n/\sqrt{\log n}=\infty \quad and \quad b_n=o\big(n^{1/3}\big) \qquad (n\to\infty).$$

We now comment on the assumption on b_n. By the fact that $R_n\le n$, the assumption $b_n=o(n^{1/3})$ is best possible for (8.5.8) in the case $d=3$. When $d=2$, by the fact that

$$\mathbb{E}R_n-R_n\le\mathbb{E}R_n=O\Big(\frac{n}{\log n}\Big) \qquad (n\to\infty)$$

our condition $b_n=o(\log n)$ is best possible for (8.5.7).

Most of the proof of Theorem 8.5.3 is essentially same as the ones for Theorem 8.2.1 and Theorem 8.3.1: In view of Theorem 8.1.3 (with $p=2$), we use the random quantity

$$\frac{1}{2}\sum_{x\in\mathbb{Z}^d}R^2(n,x,\epsilon)$$

to approximate $\mathbb{E}R_n - R_n$ exponentially. Every step appears as a obvious modification of the correspondent argument used in the proof of Theorem 8.2.1 and Theorem 8.3.1.

The only exception is the part correspondent to Lemma 8.2.2. To complete the argument for Theorem 8.5.3, we prove the following Lemma 8.5.9 in the rest of the section.

To begin, recall that

$$J_n = \#\{S[1,n] \cap \widetilde{S}[1,n]\}$$

and write, for $\epsilon > 0$,

$$J_n(\epsilon) = \sum_{x \in \mathbb{Z}^d} R(n,x,\epsilon)\widetilde{R}(n,x,\epsilon)$$

where $R(n,x,\epsilon)$ is defined in (8.1.4) and $\widetilde{R}(n,x,\epsilon)$ is generated by $\{\widetilde{S}(k)\}$ in a way same as how $R(n,x,\epsilon)$ is generated by $\{S(k)\}$.

Lemma 8.5.4 *(1). When $d = 2$,*

$$\lim_{\epsilon \to 0^+} \limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta \sqrt{\frac{b_n}{n}} \log n |J_n - J_n(\epsilon)|^{1/2}\right\} = 0 \qquad (8.5.9)$$

for any $\theta > 0$ and for any positive sequence b_n satisfying

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o(\log n) \qquad (n \to \infty).$$

(2). When $d = 3$,

$$\lim_{\epsilon \to 0^+} \limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta \Big(\frac{b_n}{n}\Big)^{1/4} |J_n - J_n(\epsilon)|^{1/2}\right\} = 0 \qquad (8.5.10)$$

for any $\theta > 0$ and for any positive sequence b_n satisfying

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o(n^{1/3}) \qquad (n \to \infty).$$

Proof. For similarity we only deal with the case $d = 2$. The challenge we face here is much more substantial than the proof of Lemma 8.2.2. In contrary to (6.1.13) (which is used in the proof of Lemma 8.2.2), there is no the moment inequality providing bounds directly for $J_n - J_n(\epsilon)$.

Let $0 = n_0 < n_1 < \cdots < n_{[b_n]} = n$ be an integer partition of the interval $[0, n]$ such that

$$n_k - n_{k-1} = [n/b_n] \text{ or } [n/b_n] \pm 1 \qquad k = 1, \cdots [b_n].$$

Recall that

$$R(n,x,\epsilon)=\Big(\frac{b_n}{n}\Big)\det(\Gamma)^{-1/2}\sum_{y\in S[1,n]}p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}(y-x)\Big).$$

Write

$$\xi(n,x,\epsilon)=\Big(\frac{b_n}{n}\Big)\det(\Gamma)^{-1/2}\sum_{j=1}^{[b_n]}\sum_{y\in S(n_{j-1},n_j]}p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}(y-x)\Big).$$

Let $h_{n,\epsilon}$ be a function on $\mathbb{Z}^2$ defined by

$$h_{n,\epsilon}(x)=\Big(\frac{b_n}{n}\Big)\det(\Gamma)^{-1}\sum_{y\in\mathbb{Z}^2}p_\epsilon\Big(x-\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}y\Big)p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}y\Big).$$

In the risk of over-simplification of the notation, we define

$$R(x)=1_{\{x\in S[1,n]\}},$$

$$\xi(x)=\sum_{j=1}^{[b_n]}1_{\{x\in S_j(n_{j-1},n_j]\}},$$

and

$$R_\epsilon(x)=\sum_{y\in S[1,n]}h_{n,\epsilon}\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}(y-x)\Big),$$

$$\xi_\epsilon(x)=\sum_{j=1}^{[b_n]}\sum_{y\in S(n_{j-1},n_j]}h_{n,\epsilon}\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}(y-x)\Big).$$

We define $\widetilde{R}(n,x,\epsilon)$, $\widetilde{\xi}(n,x,\epsilon)$, $\widetilde{R}(x)$, $\widetilde{\xi}(x)$, $\widetilde{R}_\epsilon(x)$, $\widetilde{\xi}_\epsilon(x)$ in an obvious way.

Observe that

$$J_n(\epsilon)=\sum_{x\in\mathbb{Z}^2}R(n,x,\epsilon)\widetilde{R}(n,x,\epsilon)=\sum_{x\in Z^2}1_{\{x\in S[1,n]\}}\widetilde{R}_\epsilon(x).$$

With the notation used in the Hilbert space $l^2(\mathbb{Z}^2)$, therefore

$$J_n-J_n(\epsilon)=\langle R,\widetilde{R}\rangle-\langle R,\widetilde{R}_\epsilon\rangle=\langle R,\widetilde{R}-\widetilde{R}_\epsilon\rangle.$$

Hence,

$$\big|J_n-J_n(\epsilon)\big|\le\big|\langle\xi,\widetilde{R}-\widetilde{R}_\epsilon\rangle\big|+\big|\langle\xi-R,\widetilde{R}-\widetilde{R}_\epsilon\rangle\big|.$$

For the first term on the right hand side,

$$\langle \xi, \widetilde{R} - \widetilde{R}_\epsilon \rangle = \langle \xi, \widetilde{R} \rangle - \langle \xi, \widetilde{R}_\epsilon \rangle = \langle \xi, \widetilde{R} \rangle - \langle \xi_\epsilon, \widetilde{R} \rangle = \langle \xi - \xi_\epsilon, \widetilde{R} \rangle.$$

As for the second term, observe that $\xi(x) - R(x) \geq 0$. Therefore,

$$\begin{aligned} \big|\langle \xi - R, \widetilde{R} - \widetilde{R}_\epsilon \rangle\big| &\leq \langle \xi - R, \widetilde{\xi} \rangle + \langle \xi - R, \widetilde{\xi}_\epsilon \rangle \\ &= \langle \xi - R, \widetilde{\xi} - \widetilde{\xi}_\epsilon \rangle + 2\langle \xi - R, \widetilde{\xi}_\epsilon \rangle \\ &= \langle \xi - R, \widetilde{\xi} - \widetilde{\xi}_\epsilon \rangle + 2\langle \xi_\epsilon - R_\epsilon, \widetilde{\xi} \rangle. \end{aligned}$$

In summary,

$$\big|J_n - J_n(\epsilon)\big| \leq \big|\langle \xi - \xi_\epsilon, \widetilde{R} \rangle\big| + \langle \xi - R, \widetilde{\xi} - \widetilde{\xi}_\epsilon \rangle + 2\langle \xi_\epsilon - R_\epsilon, \widetilde{\xi} \rangle.$$

Notice that

$$\langle \xi - R, \widetilde{\xi} - \widetilde{\xi}_\epsilon \rangle \stackrel{d}{=} \langle \xi - \xi_\epsilon, \widetilde{\xi} - \widetilde{R} \rangle \leq \big|\langle \xi - \xi_\epsilon, \widetilde{\xi} \rangle\big| + \big|\langle \xi - \xi_\epsilon, \widetilde{R} \rangle\big|.$$

For (8.5.9) to hold, therefore, we need only to show that for any $\theta > 0$,

$$\lim_{\epsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta \sqrt{\frac{b_n}{n}} \log n \big|\langle \xi - \xi_\epsilon, \widetilde{R} \rangle\big|^{1/2}\right\} = 0 \tag{8.5.11}$$

$$\lim_{\epsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta \sqrt{\frac{b_n}{n}} \log n \big|\langle \xi - \xi_\epsilon, \widetilde{\xi} \rangle\big|^{1/2}\right\} = 0 \tag{8.5.12}$$

and

$$\lim_{n \to \infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta \sqrt{\frac{b_n}{n}} \log n \langle \xi_\epsilon - R_\epsilon, \widetilde{\xi} \rangle^{1/2}\right\} = 0 \tag{8.5.13}$$

for any fixed $\epsilon > 0$.

Observe that for any integer $m \geq 1$,

$$\begin{aligned} &\mathbb{E}\langle \xi - \xi_\epsilon, \widetilde{R} \rangle^m \\ &= \sum_{x_1, \cdots, x_m \in \mathbb{Z}^2} \left[\mathbb{E} \prod_{k=1}^m \Big(\xi(x_k) - \xi_\epsilon(x_k)\Big)\right] \left[\mathbb{E} \prod_{k=1}^m 1_{\{x_k \in S[1,n]\}}\right] \\ &\leq \left\{\sum_{x_1, \cdots, x_m \in \mathbb{Z}^2} \left[\mathbb{E} \prod_{k=1}^m \Big(\xi(x_k) - \xi_\epsilon(x_k)\Big)\right]^2\right\}^{1/2} \\ &\times \left\{\sum_{x_1, \cdots, x_m \in \mathbb{Z}^2} \left[\mathbb{E} \prod_{k=1}^m 1_{\{x_k \in S[1,n]\}}\right]^2\right\}^{1/2} \\ &= \Big\{\mathbb{E}\langle \xi - \xi_\epsilon, \widetilde{\xi} - \widetilde{\xi}_\epsilon \rangle^m\Big\}^{1/2} \Big\{\mathbb{E} J_n^m\Big\}^{1/2}. \end{aligned}$$

For (8.5.11) to hold, it is sufficient to show that

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m \times\bigg[\Big\{\mathbb{E}\langle\xi-\xi_\epsilon,\widetilde{\xi}-\widetilde{\xi}_\epsilon\rangle^m\Big\}\Big\{\mathbb{E}J_n^m\Big\}\bigg]^{1/4}=0. \tag{8.5.14}$$

By Cauchy-Schwartz inequality, for any $\lambda>0$

$$\begin{aligned}&\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m\bigg[\Big\{\mathbb{E}\langle\xi-\xi_\epsilon,\widetilde{\xi}-\widetilde{\xi}_\epsilon\rangle^m\Big\}\Big\{\mathbb{E}J_n^m\Big\}\bigg]^{1/4}\\ &\le\bigg\{\sum_{m=0}^\infty\frac{(\lambda\theta)^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m\Big\{\mathbb{E}\langle\xi-\xi_\epsilon,\widetilde{\xi}-\widetilde{\xi}_\epsilon\rangle^m\Big\}^{1/2}\bigg\}^{1/2}\\ &\times\bigg\{\sum_{m=0}^\infty\frac{(\lambda^{-1}\theta)^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m\Big\{\mathbb{E}J_n^m\Big\}^{1/2}\bigg\}^{1/2}.\end{aligned}$$

Taking $p=2$ in (7.2.7), we have the bound

$$\lim_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty\frac{(\lambda^{-1}\theta)^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m\Big\{\mathbb{E}J_n^m\Big\}^{1/2}=C(\lambda^{-1}\theta)^2 \tag{8.5.15}$$

for a universal constant $C>0$. One can make the right hand side arbitrarily small by making λ sufficiently large. To establish (8.5.14), therefore, we only need show that

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m\Big\{\mathbb{E}\langle\xi-\xi_\epsilon,\widetilde{\xi}-\widetilde{\xi}_\epsilon\rangle^m\Big\}^{1/2}=0$$

for every $\theta>0$.

Indeed, this follows from an argument similar to the one used in the proof of Lemma 8.2.2. We describe it here briefly.

Write

$$\xi'_\epsilon=\lambda_{n,\epsilon}^{-2}\xi_\epsilon$$

where we choose $\lambda_{n,\epsilon}$ to the function

$$\bar{h}_{n,\epsilon}(x)=\lambda_{n,\epsilon}^{-1}h_{n,\epsilon}(x)\qquad x\in\mathbb{Z}^2$$

a probability density on $\mathbb{Z}^2$. It is easy to see that for each $\epsilon>0$ $\lambda_{n,\epsilon}\to\infty$

as $n \to \infty$. Therefore, our claim is equivalent to

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^\infty \frac{\theta^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m \times\Big\{\mathbb{E}\langle \xi-\xi_\epsilon', \widetilde{\xi}-\widetilde{\xi}'_\epsilon\rangle^m\Big\}^{1/2} = 0. \tag{8.5.16}$$

Take the additive functional

$$\mathcal{F}_k^x(S) = \sum_{j=1}^k 1_{\{x\in S(n_{j-1},n_j]\}} \qquad k = 1, \cdots, [b_n]$$

on the increasing sequence $n_1 < \cdots < b_{[b_n]}$ and take the probability density $h(x) = \bar{h}_{n,\epsilon}(x)$ in the inequality (6.1.16) of Theorem 6.1.2.

$$\begin{aligned}&\sum_{m=0}^\infty \frac{\theta^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m\Big\{\mathbb{E}\langle \xi-\xi_\epsilon', \widetilde{\xi}-\widetilde{\xi}'_\epsilon\rangle^m\Big\}^{1/2}\\ &\le \prod_{k=1}^{[b_n]}\sum_{m=0}^\infty \frac{\theta^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m\Big\{\mathbb{E}\big[\bar{D}^{(2)}_{n_k-n_{k-1},\epsilon}\big]^m\Big\}^{1/2},\end{aligned}$$

where, in connection to (5.3.32) with $d = p = 2$,

$$\begin{aligned}\bar{D}^{(2)}_{n_k-n_{k-1},\epsilon} &= \sum_{x\in\mathbb{Z}^2}\Big[1_{\{x\in S[1,n_k-n_{k-1}]\}} - \sum_{z\in S[1,n_k-n_{k-1}]}\bar{h}_{n,\epsilon}(z-x)\Big]\\ &\quad\times\Big[1_{\{x\in \widetilde{S}[1,n_k-n_{k-1}]\}} - \sum_{z\in \widetilde{S}[1,n_k-n_{k-1}]}\bar{h}_{n,\epsilon}(z-x)\Big]\\ &= \big(1+o(1)\big)D_{n_k-n_{k-1},2\epsilon} \qquad (n\to\infty).\end{aligned}$$

By (5.3.33) in Theorem 5.3.5 (with $p = 2$ and n being replaced by $n_k - n_{k-1}$),

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\Big(\frac{b_n}{n}\Big)^2(\log n)^4\mathbb{E}\big[\bar{D}^{(2)}_{n_k-n_{k-1},\epsilon}\big]^2 = 0$$

uniformly over $1 \le k \le [b_n]$. Consequently,

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\sum_{m=0}^\infty \frac{\theta^m}{m!}\Big(\sqrt{\frac{b_n}{n}}\log n\Big)^m\Big\{\mathbb{E}\big[\bar{D}^{(2)}_{n_k-n_{k-1},\epsilon}\big]^m\Big\}^{1/2} = 1$$

uniformly over $1 \le k \le [b_n]$. This leads to the requested (8.5.16) and therefore to (8.5.11).

Observe that by the inequality (6.1.12) with $a = [b_n]$ and by (5.3.26) in Theorem 5.3.4 (with n being replaced by $[n/b_n]$), the bound given in (8.5.15) remains true even when J_n is replaced by $\mathcal{J}_n \equiv \langle \xi, \widetilde{\xi} \rangle$. Consequently, (8.5.12) follows from the same argument as (8.5.11).

It remains to prove (8.5.13). Notice that

$$\langle \xi_\epsilon - R_\epsilon, \widetilde{\xi} \rangle = \langle \xi_\epsilon - R_\epsilon, \widetilde{R} \rangle + \langle \xi_\epsilon - R_\epsilon, \widetilde{\xi} - \widetilde{R} \rangle$$

and that

$$\langle \xi_\epsilon - R_\epsilon, \widetilde{\xi} - \widetilde{R} \rangle = \sum_{x \in \mathbb{Z}^2} \big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big]\big[\widetilde{\xi}(n,x,\epsilon) - \widetilde{R}(n,x,\epsilon)\big].$$

Consequently,

$$\begin{aligned} \langle \xi_\epsilon - R_\epsilon, \widetilde{\xi} - \widetilde{R} \rangle^{1/2} \le \frac{1}{2} \Big\{ \sum_{x \in \mathbb{Z}^2} \big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big]^2 \Big\}^{1/2} \\ + \frac{1}{2} \Big\{ \sum_{x \in \mathbb{Z}^2} \big[\widetilde{\xi}(n,x,\epsilon) - \widetilde{R}(n,x,\epsilon)\big]^2 \Big\}^{1/2}. \end{aligned}$$

In addition,

$$\begin{aligned} \langle \xi_\epsilon - R_\epsilon, \widetilde{R} \rangle &= \sum_{x \in \mathbb{Z}^2} \big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big] \widetilde{R}(n,x,\epsilon) \\ &\le \Big\{ \sum_{x \in \mathbb{Z}^2} \big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big]^2 \Big\}^{1/2} \Big\{ \sum_{x \in \mathbb{Z}^2} \widetilde{R}^2(n,x,\epsilon) \Big\}^{1/2}. \end{aligned}$$

Hence, for any fixed but arbitrary $\lambda > 0$

$$\begin{aligned} &\langle \mathcal{R}_\epsilon - R_\epsilon, \widetilde{R} \rangle^{1/2} \\ &\le \frac{\lambda}{2} \sum_{x \in \mathbb{Z}^2} \big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big]^2 \Big\}^{1/2} + \frac{\lambda^{-1}}{2} \Big\{ \sum_{x \in \mathbb{Z}^2} \widetilde{R}^2(n,x,\epsilon) \Big\}^{1/2}. \end{aligned}$$

Further notice that

$$\sum_{x \in \mathbb{Z}^2} \big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big]^2 \overset{d}{=} \sum_{x \in \mathbb{Z}^2} \big[\widetilde{\xi}(n,x,\epsilon) - \widetilde{R}(n,x,\epsilon)\big]^2,$$

$$\sum_{x \in \mathbb{Z}^2} \widetilde{R}^2(n,x,\epsilon) \overset{d}{=} \sum_{x \in \mathbb{Z}^2} R^2(n,x,\epsilon)$$

and that by Theorem 8.1.3 (with $d = p = 2$),

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta\sqrt{\frac{b_n}{n}} \log n \Big(\sum_{x\in\mathbb{Z}^2} R^2(n,x,\epsilon)\Big)^{1/2}\right\}\\
&= \sup_{g\in\mathcal{F}_2} \left\{2\pi \det(\Gamma)^{1/4}\Big(\int_{\mathbb{R}^2} |g(x,\epsilon)|^4 dx\Big)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^2} |\nabla g(x)|^2 dx\right\}\\
&\le \sup_{g\in\mathcal{F}_2} \left\{2\pi \det(\Gamma)^{1/4}\Big(\int_{\mathbb{R}^2} |g(x)|^4 dx\Big)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^2} |\nabla g(x)|^2 dx\right\}\\
&= 2\pi^2\sqrt{\det(\Gamma)}\kappa(2,2)^4\theta^2
\end{aligned}$$

for every $\theta > 0$ and $\epsilon > 0$, where the second step follows from Jensen inequality and the last step from Theorem C.1 in Appendix.

For (8.5.13) to hold, therefore, it is sufficient to establish that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\left\{\theta\sqrt{\frac{b_n}{n}} \log n \right.\\
&\quad\times \left.\Big(\sum_{x\in\mathbb{Z}^2} \big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big]^2\Big)^{1/2}\right\} = 0
\end{aligned} \tag{8.5.17}$$

for every fixed $\theta > 0$, $\epsilon > 0$.

Let $f \in l^2(\mathbb{Z}^2)$ satisfy $||f||_2 = 1$. We have

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^2} f(x)\big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big]\\
&= \Big(\frac{b_n}{n}\Big)\det(\Gamma)^{-1/2}\sum_{x\in\mathbb{Z}^2} T_{n,\epsilon}f(x)\Big\{\sum_{j=1}^{[b_n]} 1_{\{x\in S(n_{j-1},n_j]\}} - 1\{x\in S[1,n]\}\Big\}
\end{aligned}$$

where, by Cauchy-Schwartz inequality,

$$\begin{aligned}
T_{n,\epsilon}f(x) &\equiv \sum_{y\in\mathbb{Z}^2} f(y)p_\epsilon\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}(x-y)\Big)\\
&\le \Big\{\sum_{y\in\mathbb{Z}^2} p_\epsilon^2\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}(x-y)\Big)\Big\}^{1/2}\\
&= \Big\{\sum_{y\in\mathbb{Z}^2} p_\epsilon^2\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}y\Big)\Big\}^{1/2} \qquad x\in\mathbb{Z}^2
\end{aligned}$$

where the last step follows from translation invariance. Write

$$K_{n,\epsilon} = \Big\{\sum_{y\in\mathbb{Z}^2} p_\epsilon^2\Big(\sqrt{\frac{b_n}{n}}\Gamma^{-1/2}y\Big)\Big\}^{1/2}.$$

By the fact that

$$\sum_{j=1}^{[b_n]} 1_{\{x\in S(n_{j-1},n_j]\}} - 1\{x \in S[1,n]\} \geq 0 \qquad x \in \mathbb{Z}^2,$$

we obtain

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^2} f(x)\big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big] \\
&\leq \Big(\frac{b_n}{n}\Big)\det(\Gamma)^{-1/2}\sum_{x\in\mathbb{Z}^2} K_{n,\epsilon}\Big\{\sum_{j=1}^{[b_n]} 1_{\{x\in S(n_{j-1},n_j]\}} - 1_{\{x\in S[1,n]\}}\Big\} \\
&= \Big(\frac{b_n}{n}\Big)\det(\Gamma)^{-1/2} K_{n,\epsilon}\Big\{\sum_{j=1}^{[b_n]} \#\big\{S(n_{j-1},n_j]\big\} - \#\big\{S[1,n]\big\}\Big\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\Big(\sum_{x\in\mathbb{Z}^2}\big[\xi(n,x,\epsilon) - R(n,x,\epsilon)\big]^2\Big)^{1/2} \\
&\leq \Big(\frac{b_n}{n}\Big)\det(\Gamma)^{-1/2} K_{n,\epsilon}\Big\{\sum_{j=1}^{[b_n]} \#\big\{S(n_{j-1},n_j]\big\} - \#\big\{S[1,n]\big\}\Big\}.
\end{aligned}$$

In addition,

$$\lim_{n\to\infty}\Big(\frac{b_n}{n}\Big)\det(\Gamma)^{-1/2} K_{n,\epsilon}^2 = \int_{\mathbb{R}^2} p_\epsilon^2(x)dx = p_{2\epsilon}(0) = \frac{1}{4\pi\epsilon}.$$

Therefore, the requested (8.5.17) follows from Part (1) of Lemma 7.1.4. □

8.6 Laws of the iterated logarithm

Among the large deviations established in this chapter, those including $b_n = \log\log n$ apply to the law of the iterated logarithm. The following theorem appears as a consequence of Theorem 8.1.1.

Theorem 8.6.1 *For $d = 1$ and $p > 1$,*

$$\limsup_{n\to\infty} n^{-\frac{p+1}{2}}(\log\log n)^{-\frac{p-1}{2}} \sum_{x\in\mathbb{Z}} l^p(n,x)$$
$$= 2^{\frac{p+1}{2}}\sigma^{-(p-1)}(p-1)^{\frac{p-1}{2}}(p+1)^{\frac{p-3}{2}} B\Big(\frac{1}{p-1}, \frac{1}{2}\Big)^{-(p-1)} \qquad a.s.$$

Proof. The proof follows from an obvious modification of the proof of Theorem 4.4.1. □

Corespondent to Theorem 8.2.1, Theorem 8.2.3, Theorem 8.5.1 and (8.5.7) in Theorem 8.5.3, we have[4]

Theorem 8.6.2 *Assume $d = 2$.*

$$\limsup_{n\to\infty} \frac{1}{n\log\log n}(Q_n - \mathbb{E}Q_n) = \det(\Gamma)^{-1/2}\kappa(2,2)^4 \qquad a.s. \tag{8.6.1}$$

$$\liminf_{n\to\infty} \frac{1}{n\log\log\log n}(Q_n - \mathbb{E}Q_n) = -(2\pi)^{-1}\det(\Gamma)^{-1/2} \qquad a.s. \tag{8.6.2}$$

$$\liminf_{n\to\infty} \frac{(\log n)^2}{n\log\log n}(R_n - \mathbb{E}R_n) = -(2\pi)^{-2}\sqrt{\det(\Gamma)}\kappa(2,2)^4 \qquad a.s. \tag{8.6.3}$$

$$\limsup_{n\to\infty} \frac{(\log n)^2}{n\log\log\log n}(R_n - \mathbb{E}R_n) = -(2\pi)\sqrt{\det(\Gamma)} \qquad a.s. \tag{8.6.4}$$

Proof. Due to similarity we only prove (8.6.1) and (8.6.2). By examining the proof of Theorem 4.4.1 carefully, one can see that the only thing that needs to be proved is that

$$\sup_{n\geq 1}\mathbb{E}\exp\Big\{\frac{\theta}{n}\max_{1\leq k\leq n}|Q_k - \mathbb{E}Q_k|\Big\} < \infty \tag{8.6.5}$$

for some $\theta > 0$. This can be done by a argument used in the proof of Theorem 6.4.3 □

In the case $d = 3$, the large deviations given in Theorem 8.3.1 and in Theorem 8.5.2 lead to the following theorem.

[4]Without the assumption on $(2+\delta)$-moment in Theorem 8.2.3 and in Theorem 8.5.1, one still can have some weaker versions of large deviations leading to (8.6.2) and (8.6.4), we refer the interested reader to Theorem 1.1 in [7]

Theorem 8.6.3 *Let $d = 3$.*

$$\limsup_{n\to\infty} \frac{\pm(Q_n - \mathbb{E}Q_n)}{\sqrt{n\log n\log\log n}} = \frac{1}{\pi\sqrt{\det(\Gamma)}} \qquad a.s. \tag{8.6.6}$$

$$\limsup_{n\to\infty} \frac{\pm(R_n - \mathbb{E}R_n)}{\sqrt{n\log n\log\log n}} = \frac{\gamma_S}{\pi\sqrt{\det(\Gamma)}} \qquad a.s. \tag{8.6.7}$$

Proof. Due to similarity we only prove (8.6.6). Using the integrability established in (6.4.14), Theorem 6.4.3 instead of (6.4.1) in Theorem 6.4.1, a slight modification of the estimate conducted in the proof of Theorem 8.3.1 gives that

$$\limsup_{n\to\infty} \frac{1}{b_n}\log\mathbb{P}\Big\{\max_{1\le k\le n}|Q_k - \mathbb{E}Q_k| \ge \lambda\sqrt{nb_n\log n}\Big\} \le -2$$

for sufficiently large $\lambda > 0$. Taking $b_n = \log\log n$, by Theorem 8.3.1 and by a Borel-Cantelli-type argument (see the proof of of Theorem 4.4.1) we have (8.6.6). $\square$

In addition to the results established in the dimensions $d \le 3$, the laws of the iterated logarithm for $Q_n - \mathbb{E}Q_n$ and for $R_n - \mathbb{E}R_n$ were studied in the early days in the case $d \ge 4$. The following theorem essentially belongs to Jain and Pruitt ([96]).[5]

Theorem 8.6.4 *Let $d \ge 4$.*

$$\limsup_{n\to\infty} \frac{\pm(Q_n - \mathbb{E}Q_n)}{\sqrt{n\log\log n}} = \sqrt{2}\mathfrak{D} \qquad a.s. \tag{8.6.8}$$

$$\limsup_{n\to\infty} \frac{\pm(R_n - \mathbb{E}R_n)}{\sqrt{n\log\log n}} = \sqrt{2}\gamma_S^2\mathfrak{D} \qquad a.s. \tag{8.6.9}$$

where the constant $\mathfrak{D} > 0$ is given in Lemma 5.5.2.

We omit the proof for which we refer an interested reader to [96]. Instead we point out the fact the the main part of the proof relies on some large deviation tail estimates in the special case $b_n = \log\log n$.[6] The forms of these large deviations will be discussed in the next section.

[5] Jain and Pruitt only proved (8.6.9), but their argument can be easily applied to the proof of (8.6.8).

[6] Jain and Pruitt did not formulate them explicitly into the form of large deviations.

8.7 What do we expect in $d \geq 4$?

In the light of the central limit theorem given in Theorem 5.5.3, it is natural to have a large deviation principle with Gaussian tail when $d \geq 4$, at least for the small deviation scale b_n. As mentioned in the end of prior section, this is proved to be the case for $b_n = \log\log n$. For the full spectrum of b_n responding to the large deviation with Gaussian tail, here is our conjecture.

Conjecture 8.7.1 *In the case $d \geq 4$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \pm (Q_n - \mathbb{E}Q_n) \geq \lambda\sqrt{nb_n}\Big\} = -\frac{\lambda^2}{2\mathfrak{D}^2} \tag{8.7.1}$$

for any $\lambda > 0$ and for any positive sequence b_n satisfying

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o(n^{1/3}) \qquad (n \to \infty);$$

and

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \pm (R_n - \mathbb{E}R_n) \geq \lambda\sqrt{nb_n}\Big\} = -\frac{\lambda^2}{2\gamma_S^2\mathfrak{D}^2} \tag{8.7.2}$$

for any $\lambda > 0$ and for any positive sequence b_n satisfying

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o\Big(n^{\frac{d-2}{d+2}}\Big) \qquad (n \to \infty)$$

where constant $\mathfrak{D} > 0$ is given in Lemma 5.5.2.

Here is what we can do in connection to the large deviations with Gaussian tails: By an argument same as the one used for Theorem 8.3.2 and for (8.5.5) in Theorem 8.5.2 we obtain the following bounds of sub-Gaussian tails.

Theorem 8.7.2 *When $d \geq 4$,*

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ Q_n - \mathbb{E}Q_n \leq -\lambda\sqrt{nb_n}\Big\} \leq -\frac{\lambda^2}{2\mathfrak{D}^2}, \tag{8.7.3}$$

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ R_n - \mathbb{E}R_n \geq \lambda\sqrt{nb_n}\Big\} \leq -\frac{\lambda^2}{2\gamma_S^2\mathfrak{D}^2} \tag{8.7.4}$$

for $\lambda > 0$ and for any positive sequence b_n satisfying

$$b_n \longrightarrow \infty \quad \textit{and} \quad b_n = o(n) \qquad (n \to \infty).$$

Again, we conjecture that the sub-Gaussian tails can be strengthened into Gaussian tails.

Our major concern is on the possibility of the large deviations with non-Gaussian tails in the case $d \geq 4$. When $d \geq 5$, Asselah ([3]) obtained a non-Gaussian large deviation principle for $Q_n - \mathbb{E}Q_n$ which states that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}\big\{Q_n - \mathbb{E}Q_n \geq \lambda n\big\} = -\sqrt{2\lambda}\mathcal{C}(S). \tag{8.7.5}$$

Here we adapt the notation used in Theorem 7.5.3:

$$\mathcal{C}(S) = \inf\big\{||h||_2; \quad h \geq 0 \text{ and } ||\mathfrak{K}_h|| \geq 1\big\}$$

where for any non-negative $h \in l^2(\mathbb{Z}^d)$, $\mathfrak{K}_h$ is the continuous self-adjoint operator defined by

$$\mathfrak{K}_h(g)(x) = \sqrt{e^{h(x)}-1}\sum_{y\in\mathbb{Z}^d} \mathfrak{G}(y-x)g(y)\sqrt{e^{h(y)}-1} \qquad g \in l^2(\mathbb{Z}^d)$$

and

$$\mathfrak{G}(x) = \sum_{k=1}^{\infty} \mathbb{P}\{S(k) = x\} \qquad x \in \mathbb{Z}^d$$

is the Green's function of the random walk. One can see that (8.7.5) is related to Theorem 7.5.3 with $b_n = \sqrt{n}$. A natural question is to find a full spectrum of b_n such that (8.7.5) holds.

Another problem is on the non-Gaussian large deviations for $R_n - \mathbb{E}R_n$ in the case $d \geq 5$. A related work by van den Berg, Bolthausen and den Hollander ([11]), which shall be discussed in the "Notes and comments" section below.

A comparison between (8.7.5) and (8.8.2) (in the "Notes and comments" section) for $d \geq 5$ indicates a drastically difference between the large deviations (non-Gaussian tails) for $Q_n - \mathbb{E}Q_n$ and $R_n - \mathbb{E}R_n$.

Conjecture 8.7.3 *When $d \geq 5$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{Q_n - \mathbb{E}Q_n \geq \lambda b_n^2\big\} = -\sqrt{2\lambda}\mathcal{C}(S) \tag{8.7.6}$$

for any $\lambda > 0$ and for any b_n satisfying

$$b_n/\sqrt[3]{n} \longrightarrow \infty \quad and \quad b_n = o(n) \qquad (n \to \infty);$$

and

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{R_n - \mathbb{E}R_n \leq -\lambda b_n^{\frac{d}{d-2}}\Big\} = \widetilde{\mathcal{C}}(S)\lambda^{\frac{d-2}{d}} \tag{8.7.7}$$

for any $\lambda > 0$ *and for any* b_n *satisfying*

$$b_n/n^{\frac{d-2}{d+2}} \longrightarrow \infty \quad \textit{and} \quad b_n = o\Big(n^{\frac{d-2}{d}}\Big) \qquad (n \to \infty).$$

The tail behaviors of $Q_n - \mathbb{E}Q_n$ and $R_n - \mathbb{E}R_n$ are closely related to those of, respectively, I_∞ and J_∞. In particular, the reason why $Q_n - \mathbb{E}Q_n$ and $R_n - \mathbb{E}R_n$ behave differently in the super-critical dimension $d \geq 5$ was explained in the end of section 7.5. If Conjecture 8.7.3 holds, a natural problem is to identify the constant $\widetilde{\mathcal{C}}(S)$. We believe that the value of $\widetilde{\mathcal{C}}(S)$ is related to that of $\widetilde{\mathcal{C}}(d,p)$ (with $p = 2$) given in Conjecture 7.5.7.

The extension from (8.7.5) to (8.7.6) can be very delicate. Indeed, by the fact that $\mathbb{E}Q_n = O(n)$, the large deviation in (8.7.5) is essentially a story of the tail behavior of Q_n. When $b_n = o(\sqrt{n})$, on the other hand, the renormalization makes the model much less tractable. In addition, $Q_n - \mathbb{E}Q_n$ is no longer approximatable by its smoothed version in the super-critical dimension $d \geq 5$.

Perhaps the dimension $d = 4$ is the most interesting and most challenging case. Nothing significant regarding to the large deviations for $Q_n - \mathbb{E}Q_n$ has been known in the case $d = 4$, especially the one with non-Gaussian tail. On the analytic side, the Gagliardo-Nirenberg inequality given in (3.3.2) still holds in the form of the Sobolev inequality

$$||f||_4 \leq C||\nabla f||_2 \qquad f \in W^{1,2}(\mathbb{R}^4) \tag{8.7.8}$$

when $d = 4$ and $p = 2$. On the other hand, the L_2-norm $||f||_2$ is no longer relevant. All of these indicate that in the sense of the non-Gaussian tail large deviations for $Q_n - \mathbb{E}Q_n$, the case $d = 4$ is critical between a diffusive behavior pattern (when $d \leq 3$) and a non-diffusive behavior pattern (when $d \geq 5$) of the random walks.

Conjecture 8.7.4 *When* $d = 4$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{Q_n - \mathbb{E}Q_n \geq \lambda b_n^2\big\} = -\frac{\sqrt{\lambda}}{4\pi} \det(\Gamma)^{-1/4} \kappa(4,2)^{-2} \tag{8.7.9}$$

for any $\lambda > 0$ *and for any positive sequence* b_n *satisfying*

$$b_n/n^{1/3} \longrightarrow \infty \quad \textit{and} \quad b_n = o(n) \qquad (n \to \infty)$$

and,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{R_n - \mathbb{E}R_n \leq -\lambda b_n^2\big\} = -\frac{\sqrt{\lambda}}{4\pi\gamma_S} \det(\Gamma)^{-1/4} \kappa(4,2)^{-2} \tag{8.7.10}$$

for any $\lambda > 0$ *and for any positive sequence* b_n *satisfying*

$$b_n/n^{1/3} \longrightarrow \infty \quad \textit{and} \quad b_n = o(\sqrt{n}) \qquad (n \to \infty).$$

In addition to the difficulty we face in the super-critical dimensions, the extra challenge here is the lack of sharp tail estimate for I_n and J_n in the case $d = 4$ and $p = 2$. In this sense, the final solution of Conjecture 8.7.4 depends on the outcome of Conjecture 7.4.2. Finally, the conjecture made in (8.7.10) is partially supported by van den Berg, Bolthausen and den Hollander's large deviation for given in (8.8.2) ($d = 4$) in the "Notes and comments" section below.

8.8 Notes and comments

Section 8.1

Theorem 8.1.1 is a discrete version of Theorem 4.2.1 and was established by Chen and Li ([31]). We refer the reader to [24] (Theorem 3.1,[24]) for a detail proof of Theorem 8.1.2.

Exercise 1. Prove Theorem 8.1.3.

The following exercise originates from Khoshnevisan and Lewis ([104]).

Exercise 2. Let ζ_n be the random walk in random scenery defined in (5.6.1), where we add the extra bounded assumption $|\xi(0)| \le C$ on the scenery $\{\xi(x);\ \ x \in \mathbb{Z}^d\}$. Prove that as $d = 1$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{ \pm \zeta_n \ge \lambda(nb_n)^{3/4}\big\} = -\frac{1}{2}\sigma^{2/3}(2\lambda)^{4/3} \qquad \lambda > 0$$

for any positive sequence b_n satisfying $b_n \to \infty$ with $b_n = o\big(n^{1/3}\big)$.

What can we say about the case $d \ge 2$?

Section 8.2

Theorem 8.2.2 was proved in [6] and appears as the discrete version of Theorem 4.3.1.

Exercise 3. We look for another approach to prove Theorem 8.2.1. First note that Theorem 8.2.1 can be proved in the same way as Theorem 4.3.1 if we assume additionally that

$$\mathbb{E}e^{i\lambda\cdot S(1)} \ge 0 \qquad \forall \lambda \in \mathbb{R}^2.$$

To prove Theorem 8.2.1 in the full generality, we first define a *compound Poisson process* $X(t)$. Let $\{\tau_k\}_{k\ge 1}$ be an i.i.d. sequence of exponential times such that $\mathbb{E}\tau_1 = 1$ and that $\{\tau_k\}_{k\ge 1}$ is independent of $\{S(k)\}_{k\ge 1}$. Write

$$T_0 = 0, \qquad T_k = \tau_1 + \cdots + \tau_k$$

and define

$$N_t = k \qquad \text{if } T_{k-1} \le t < T_k \qquad k = 1, 2, \cdots .$$

(1). Prove that N_t is a Poisson process with

$$\mathbb{P}\{N_t = n\} = e^{-t}\frac{t^n}{n!} \qquad n = 0, 1, 2, \cdots$$

and that the process $X(t)$ defined by

$$X(t) = S(N_t)$$

is a pure jump Lévy process on $\mathbb{Z}^2$ with

$$\mathbb{E}\exp\big\{i\lambda\cdot X(t)\big\} = \exp\big\{-t[1-\varphi(\lambda)]\big\}$$

where $\varphi(\lambda) = \mathbb{E}e^{i\lambda\cdot S(1)}$. In particular

$$\mathbb{E}\exp\big\{i\lambda\cdot X(t)\big\} > 0.$$

(2). Define

$$\mathcal{Q}_t = \iint_{\{0\le r<s\le t\}} 1_{\{X(r)=X(s)\}}drds.$$

Apply the argument used for Theorem 4.3.1 to prove that if $b_t \to \infty$ and $b_t = o(t)$ as $t \to \infty$,

$$\lim_{t\to\infty}\frac{1}{b_t}\log\mathbb{P}\big\{\mathcal{Q}_t - \mathbb{E}\mathcal{Q}_t \ge \lambda t b_t\big\} = -\lambda\sqrt{\det(\Gamma)}\kappa(2,2)^{-4} \qquad \forall\lambda > 0.$$

(3) Prove that for any $\epsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\big\{|\sqrt{\mathcal{Q}_n} - \sqrt{Q_n}| \ge \epsilon\sqrt{nb_n}\big\} = -\infty.$$

Hint: First prove that

$$\mathcal{Q}_{T_n} = \frac{1}{2}\sum_{x\in\mathbb{Z}^2}\left[\sum_{k=1}^n \tau_k 1_{\{S(k)=x\}}\right]^2.$$

Write $\omega_k = \tau_k - 1$ and then prove that for any $\theta > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_n}{n}} \left(\sum_{x\in\mathbb{Z}^2} \left[\sum_{k=1}^{n} \omega_k 1_{\{S(k)=x\}} \right]^2 \right)^{1/2} \right\} = 0.$$

(4). Prove Theorem 8.2.1 by the results from (2) and (3).

Section 8.3 and 8.4

In the light of the central limit theorem given in Theorem 5.5.3, the large deviation (also called moderate deviation) with Gaussian tail, which is stated in Theorem 8.3.1 and Theorem 8.3.2, is naturally expected.

In connection to Theorem 8.4.2, Asselah ([2]) obtained the bounds

$$\exp\left\{ -c_1\lambda^{2/3}n^{1/3}\right\} \le \mathbb{P}\left\{Q_n \ge \lambda n\right\} \le \exp\left\{ -c_2\lambda^{1/3}n^{1/3}\right\}$$

for sufficiently large $\lambda > 0$. By the fact that $\mathbb{E}Q_n = O(n)$ one can replace Q_n by $Q_n - \mathbb{E}Q_n$ in above estimate. Further, Fleischmann, Mörters and Wachtel pointed out (Proposition 11, [75]) that

$$\mathbb{P}\left\{|Q_n - \mathbb{E}Q_n| \ge \sqrt{nb_n^3}\right\} \le \exp\left\{ -cb_n\right\}$$

for $b_n \ge (\log n)^3$. These discoveries suggested a non-Gaussian type large deviations. In particular, Asselah's bounds correspond to the special case of Theorem 8.4.2 in which $b_n = n^{1/3}$.

Theorem 8.3.1 and Theorem 8.4.2 seem to suggest co-existence of Gaussian and non-Gaussian large deviations for $Q_n - \mathbb{E}Q_n$ (and for $R_n - \mathbb{E}R_n$ later) as $d \ge 3$, and the relevance of the Gagliardo-Nirenberg inequality up to dimension 4, within which the inequality legally holds. The conjectures made in section 8.7 are partially based on this belief.

Section 8.5

An early result by Donsker and Varadhan ([56]) shows that

$$\lim_{n\to\infty} n^{-\frac{d}{d+2}} \log \mathbb{E}\exp\left\{ -\theta R_n\right\} = -\theta^{\frac{2}{2+d}} \left(\frac{d+2}{2}\right)\left(\frac{2\lambda_d}{d}\right)^{\frac{d}{d+2}}$$

where $\lambda_d > 0$ is the lowest eigenvalue of the generator $-(\Delta/2)$ on ball of unit volume with zero boundary values. See Donsker-Varadhan [54], Bolthausen [14] and Sznitman [149] for some versions of this result in the setting of

Wiener sausages. For the large deviations of different side, Hamana ([85]), Hamana and Kesten ([86], [87]) proved that for any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{R_n \geq \lambda n\}$$

yields a non-trivial limit. This is complementary to (8.5.5) (and later to 8.7.4)) by letting $b_n = n$.

Historically, the limit laws for the renormalized range $R_n - \mathbb{E}R_n$ was first investigated in high dimensions ($d \geq 3$). The central limit theorems given in Theorem 5.5.3 and the related Gaussian tails were realized in early days. The study of the large deviations were often formulated in the forms of the law of the iterated logarithm, where b_n increases in the iterated logarithmic rates — too slow for non-Gaussian tails. Consequently, the early publications concluded with Gaussian tails. Here we mention the papers by Jain and Pruitt ([96], Hamana ([84]) for $d \geq 4$ and the paper by Bass and Kumagai ([9]) for $d = 3$. Theorem 8.5.2 summarizes the Gaussian-type large deviations for $R_n - \mathbb{E}R_n$ in $d = 3$.

We mention the paper by van den Berg, Bolthausen and den Hollander ([10]), for a possible link to our non-Gaussian-type large deviations stated in Theorem 8.5.3. Recall that

$$W^\epsilon(t) = \bigcup_{0\leq s\leq t} \big\{x \in \mathbb{R}^d;\ |W(s) - x| \leq \epsilon\big\}$$

is the Wiener sausage. In[10], van den Berg, Bolthausen and den Hollander proved that for any $b > 0$,

$$\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}\Big\{|W^\epsilon(t)| \leq \frac{bt}{\log t}\Big\} = -\Psi_2(b) \tag{8.8.1}$$

when $d = 2$; and that

$$\lim_{t\to\infty} t^{-\frac{d-2}{d}} \log \mathbb{P}\big\{|W^\epsilon(t)| \leq bt\big\} = -\Psi_d(b) \tag{8.8.2}$$

for $d \geq 3$, where $\Psi_d(b)$ is given in terms of variation. Further,

$$\lim_{b\to 2\pi^-} (2\pi - b)^{-1}\Psi_2(b) = (2\pi)^{-2}\kappa(2,2)^{-4}$$

and

$$\lim_{b\to\theta_d(\epsilon)^-} \big(\theta_d(\epsilon) - b\big)^{-2/d}\Psi_d(b) = (2\pi)^{-\frac{d-2}{d}}\theta_d(\epsilon)^{-4/d}\kappa(d,2)^{-8/d}$$

for $d = 3, 4$, where $\theta_d(\epsilon)$ is defined in (7.6.4).

Theorem 8.5.1 and the "$d = 2$" part of Theorem 8.5.3 were established by Bass, Chen and Rosen ([7]), where it was required that $b_n = o\big((\log n)^{1/5}\big)$. Here we improve the later by optimizing the conditions on b_n and therefore close the gap between the van den Berg, Bolthausen and den Hollander' large deviations and the large deviations in Theorem 8.5.3.

By comparing Theorem 8.5.3 and Theorem 8.2.1, 8.4.2, one can see how $Q_n - \mathbb{E}Q_n$ and $R_n - \mathbb{E}R_n$ are related as far as the large deviations are concerned.

Section 8.6

Theorem 8.6.1 were achieved in [31]. The laws of the iterated logarithm (8.6.1) and (8.6.2) were obtained in [6]; (8.6.2) and (8.6.4) were obtained in [7]. Prior to [7], Bass and Kumagai ([9]) had established (8.6.4) without identifying the constant on the right-hand side. In the same paper, they also accomplished (8.6.7) in functional form. As mentioned before, (8.6.9) is due to Jain and Pruitt ([96]).

Section 8.7

Appendix

A Green's function

Recall that the Green's function $G(x)$ of a d-dimensional Brownian motion is defined as

$$G(x) = \int_0^\infty e^{-t} p_t(x)dx = \frac{1}{(2\pi)^{d/2}} \int_0^\infty t^{-d/2} \exp\Big\{ -\frac{1}{2}\Big(2t + \frac{|x|^2}{t}\Big)\Big\} dt.$$

The major concern is the p-square integrability of $G(\cdot)$ on $\mathbb{R}^d$. It is easy to see that this is totally determined by behavior of $G(\cdot)$ near $x = 0$.

Theorem A.1 *When* $|x| \to 0$,

$$G(x) \sim \begin{cases} \dfrac{1}{\sqrt{2}} & d = 1 \\[2ex] -\dfrac{1}{\pi} \log |x| & d = 2 \\[2ex] \dfrac{1}{2\pi^{d/2}} \Gamma\Big(\dfrac{d-2}{2}\Big)|x|^{-(d-2)} & d \geq 3. \end{cases}$$

In particular,

$$\int_{\mathbb{R}^d} G^p(x)dx < \infty$$

if and only if $p(d-2) < d$.

Proof. We first consider the case $d \geq 2$. For any $v \geq 0$, set

$$\mathcal{K}_v(\theta) = \frac{1}{2} \int_0^\infty \frac{1}{t^{v+1}} \exp\Big\{ -t - \frac{\theta}{t}\Big\} dt \qquad \theta > 0.$$

The function $\mathcal{K}_v(\theta)$ is called *Bessel-Clifford function of the second kind*, which was first introduced by Clifford ([35]). We refer to the book by Watson ([156]) for the facts about $\mathcal{K}_v(\theta)$ used here. Let $K_v(\theta)$ be the *modified Bessel function of the second kind*, which solves the *modified Bessel equation* (see [156] for detail)

$$\theta^2\frac{d^2y}{d\theta^2}+\theta\frac{dy}{d\theta}-(\theta^2+v^2)y=0.$$

The fact we use here is that

$$\mathcal{K}_v(\theta)=\theta^{-v/2}K_v(2\sqrt{\theta})\sim\begin{cases}-\dfrac{1}{2}\log\theta & v=0\\[2ex] \dfrac{\Gamma(v)}{2}\theta^{-v} & v>0\end{cases}$$

as $\theta\to 0^+$. Thus, the desired conclusion follows from the fact that

$$G(x)=\frac{2}{(2\pi)^{d/2}}\mathcal{K}_v\Big(\frac{|x|^2}{2}\Big)$$

with $v=\dfrac{d-2}{2}$.

The case $d=1$ follows from (A.1) below. □

The close form of $G(x)$ can be reached for some d. We start with the case $d=1$ and we show that

$$G(x)=\frac{1}{\sqrt{2}}e^{-\sqrt{2}|x|}\qquad x\in\mathbb{R}. \tag{A.1}$$

Indeed,

$$\begin{aligned}G(x)&=\frac{1}{\sqrt{2\pi}}\int_0^\infty t^{-1/2}\exp\Big\{-\frac{1}{2}\Big(2t+\frac{|x|^2}{t}\Big)\Big\}dt\\&\overset{u=\sqrt{2t}}{=}\frac{1}{\sqrt{\pi}}\int_0^\infty\exp\Big\{-\frac{1}{2}\Big(u^2+\frac{2|x|^2}{u^2}\Big)\Big\}du\\&=\frac{1}{\sqrt{\pi}}e^{-\sqrt{2}|x|}\int_0^\infty\exp\Big\{-\frac{1}{2}\Big(u-\frac{\sqrt{2}|x|}{u}\Big)^2\Big\}du\\&=\frac{1}{\sqrt{\pi}}e^{-\sqrt{2}|x|}I(|x|)\end{aligned}$$

where

$$
\begin{aligned}
I(|x|) &\equiv \int_0^\infty \exp\Big\{-\frac{1}{2}\Big(u-\frac{\sqrt{2}|x|}{u}\Big)^2\Big\}du\\
&= \int_0^\infty \Big(1+\frac{\sqrt{2}|x|}{u^2}\Big)\exp\Big\{-\frac{1}{2}\Big(u-\frac{\sqrt{2}|x|}{u}\Big)^2\Big\}du\\
&- \int_0^\infty \frac{\sqrt{2}|x|}{u^2}\exp\Big\{-\frac{1}{2}\Big(u-\frac{\sqrt{2}|x|}{u}\Big)^2\Big\}du\\
&\overset{v=\frac{\sqrt{2}|x|}{u}}{=} \int_{-\infty}^\infty \exp\Big\{-\frac{1}{2}u^2\Big\}du - \int_0^\infty \exp\Big\{-\frac{1}{2}\Big(v-\frac{\sqrt{2}|x|}{v}\Big)^2\Big\}du\\
&= \sqrt{2\pi} - I(|x|).
\end{aligned}
$$

Consequently, $I(|x|) = \sqrt{\pi/2}$, which leads to (A.1).

In addition, write

$$G(x) = (2\pi)^{-d/2}F_d(|x|^2)$$

where

$$F_d(\theta) = \int_0^\infty t^{-d/2}\exp\Big\{-\frac{1}{2}\Big(2t+\frac{\theta}{t}\Big)\Big\}dt \qquad \theta > 0.$$

We now claim that for any $d \ge 3$,

$$F_d(\theta) = -2F'_{d-2}(\theta). \tag{A.2}$$

Indeed, for any $\theta > 0$,

$$
\begin{aligned}
&\int_\theta^\infty F_d(u)du = \int_0^\infty t^{-d/2}\Big[\int_\theta^\infty \exp\Big\{-\frac{1}{2}\Big(2t+\frac{u}{t}\Big)\Big\}du\Big]dt\\
&= 2\int_0^\infty t^{-(d-2)/2}\exp\Big\{-\frac{1}{2}\Big(2t+\frac{\theta}{t}\Big)\Big\}dt = 2F_{d-2}(\theta).
\end{aligned}
$$

Taking derivative on the both sides proves (A.2).

In principle, by (A.1) and (A.2) one can find the close form for all Green's functions $G(x)$ when the dimension d is odd. In particular, it is straightforward to check that when $d = 3$,

$$G(x) = \frac{1}{2\pi}|x|^{-1}e^{-\sqrt{2}|x|} \qquad x \in \mathbb{R}^3. \tag{A.3}$$

B Fourier transformation

We refer the reader to the books by Edwards ([69], [70] and by Donoghue ([51]) for the general theory of Fourier transformation.

For any real function f on $\mathbb{Z}^d$ such that

$$\sum_{x\in\mathbb{Z}^d} |f(x)|^2 < \infty,$$

its *Fourier transform* is defined as

$$\widehat{f}(\lambda) = \sum_{x\in\mathbb{Z}^d} f(x)e^{i\lambda\cdot x} \qquad \lambda \in \mathbb{R}^d. \tag{B.1}$$

Here we point out that the summation on the right converges in $\mathcal{L}^2\{[-\pi,\pi]^d\}$ when λ is limited to $[-\pi,\pi]^d$ and that $\widehat{f}(\lambda)$ is periodic with period 2π.

One can view above equality as the Fourier expansion of the periodic function $\widehat{f}(\lambda)$. So the classic theory of Fourier series applies here. In particular, taking the norm square and then integrating on the both sides of (B.1), by orthogonality we obtain *Parseval identity* (known also as *Bessel identity*):

$$\sum_{x\in\mathbb{Z}^d} |f(x)|^2 = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} |\widehat{f}(\lambda)|^2 d\lambda. \tag{B.2}$$

Fourier transform defines a *isometric linear operator* isometric linear operator from $l^2(\mathbb{Z}^d)$ to $\mathcal{L}^2[-\pi,\pi]$. By linearty, Parseval identity leads to the following

$$\sum_{x\in\mathbb{Z}^d} f(x)g(x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \widehat{f}(\lambda)\overline{\widehat{g}(\lambda)}d\lambda \qquad f,g\in l^2(\mathbb{Z}^d). \tag{B.3}$$

In addition, multiplying $e^{-i\lambda\cdot x}$ and then integrating on the both side of (B.1), by orthogonality again we obtain the *Fourier inversion*

$$f(x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-i\lambda\cdot x}\widehat{f}(\lambda)d\lambda \qquad x\in\mathbb{Z}^d. \tag{B.4}$$

Further, given $f\in l^2(\mathbb{Z}^d)$ and a probability density on $\mathbb{Z}^d$, define *convolution* $f*h$ as

$$f*h(x) = \sum_{y\in\mathbb{Z}^d} f(y)h(x-y) = \sum_{y\in\mathbb{Z}^d} f(x-y)h(y) \qquad x\in\mathbb{Z}^d$$

By Jensen inequality,

$$\sum_{x\in\mathbb{Z}^d} |f*h(x)|^2 \le \sum_{x\in\mathbb{Z}^d} |f(x)|^2 < \infty.$$

It is straightforward to see that

$$\widehat{f*h}(\lambda) = \widehat{f}(\lambda)\widehat{h}(\lambda). \tag{B.5}$$

For a real function $f(x)$ on $\mathbb{R}^d$, we intend to define its Fourier transform as

$$\widehat{f}(\lambda) = \int_{\mathbb{R}^d} f(x) e^{i\lambda\cdot x} dx \qquad \lambda \in \mathbb{R}^d \tag{B.6}$$

and to establish the inversion formula

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\lambda\cdot x} \widehat{f}(\lambda) d\lambda \qquad x \in \mathbb{R}^d. \tag{B.7}$$

However, this way of stating Fourier transform and its inversion sweeps some more subtle issues under the carpet. Even under the assumption of integrability on f, the Fourier transform $\widehat{f}$ is not necessarily integrable on $\mathbb{R}^d$. In the following theorem we establish the Fourier inversion for rapidly decreasing functions. Recall that an infinitely smooth function f is said to be rapidly decreasing, if its derivatives of all orders decay at infinity faster than any negative power of $|x|$. The set of all rapidly decreasing functions is denoted by $\mathcal{S}(\mathbb{R}^d)$, which is called Schwartz space.

Theorem B.1 *For any $f \in \mathcal{S}(\mathbb{R}^d)$, the Fourier inversion (B.7) holds. More generally, for any integers $k_1, \cdots, k_d \geq 0$,*

$$\Big(\frac{\partial^{k_1+\cdots+k_d}}{\partial x_1^{k_1} \cdots x_d^{k_d}} f\Big)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \Big(\prod_{j=1}^d (-i\lambda_j)^{k_j}\Big) e^{-i\lambda\cdot x} \widehat{f}(\lambda) d\lambda \tag{B.8}$$

for all $x \in \mathbb{R}^d$. Here we adopt the notation $x = (x_1, \cdots, x_d)$ and $\lambda = (\lambda_1, \cdots, \lambda_d)$.

Proof. The argument we use here is essentially provided by Robert ([135]). By the obvious fact that $\widehat{f}$ is also rapidly decreasing, the right hand sides of (B.7) and (B.8) are well defined, and (B.7) clearly implies (B.8).

We need only to show (B.7) in the case $x = 0$, for otherwise we replace $f(\cdot)$ by $f(x + \cdot)$ in the argument below.

Given $k > 0$, notice that the function

$$F_k(z) \equiv \sum_{y\in\mathbb{Z}^d} f(z + 2ky) \qquad z \in \mathbb{R}^d$$

is well defined and periodic with period $2k$. For each $x \in \mathbb{Z}^d$,

$$\begin{aligned}
&\frac{1}{(2k)^d} \int_{[-k,k]^d} \exp\Big\{ -iz \cdot \Big(\frac{\pi x}{k}\Big)\Big\} F(z) dz \\
&= \frac{1}{(2k)^d} \int_{\mathbb{R}^d} \exp\Big\{ -iz \cdot \Big(\frac{\pi x}{k}\Big)\Big\} f(z) dz \\
&= \frac{1}{(2k)^d} \widehat{f}\Big(-\frac{\pi x}{k}\Big).
\end{aligned}$$

By Fourier expansion

$$F_k(z) = \frac{1}{(2k)^d} \sum_{x \in \mathbb{Z}^d} \widehat{f}\Big(-\frac{\pi x}{k}\Big) e^{iz \cdot x} \qquad z \in \mathbb{R}^d.$$

Taking $z = 0$, we obtain

$$\sum_{y \in \mathbb{Z}^d} f(2ky) = \frac{1}{(2k)^d} \sum_{x \in \mathbb{Z}^d} \widehat{f}\Big(\frac{\pi x}{k}\Big).$$

Letting $k \to \infty$ on the both sides leads to the desired conclusion:

$$f(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\lambda) d\lambda.$$

$\square$

Given two real rapidly decreasing functions f, g, multiplying $g(x)$ and then integrating on the both sides of (B.7), we have the *Parseval identity*

$$\int_{\mathbb{R}^d} f(x)g(x)dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\lambda)\overline{\widehat{g}(\lambda)} d\lambda. \tag{B.9}$$

The convolution $f * h$ between two functions f and h is defined as

$$f * h(x) = \int_{\mathbb{R}^d} f(y)h(x-y)dy = \int_{\mathbb{R}^d} f(x-y)h(y)dy \qquad x \in \mathbb{R}^d$$

whenever the involved integrals are well defined. When $f \in \mathcal{S}(\mathbb{R}^d)$ and h is a probability density on $\mathbb{R}^d$,

$$\widehat{f * h}(\lambda) = \widehat{f}(\lambda)\widehat{h}(\lambda). \tag{B.10}$$

In addition, by integration by parts and by (B.8),

$$\begin{aligned} &\int_{\mathbb{R}^d} \nabla f(x) \nabla g(x) dx = -\int_{\mathbb{R}^d} \Delta f(x) g(x) dx \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\lambda|^2 \widehat{f}(\lambda)\overline{\widehat{g}(\lambda)} d\lambda \qquad f, g \in \mathcal{S}(\mathbb{R}^d). \end{aligned} \tag{B.11}$$

Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{L}^2(\mathbb{R}^d)$, a standard procedure of approximation based on (B.9) extends uniquely the Fourier transform into a linear isometric operator from $\mathcal{L}^2(\mathbb{R}^d)$ to the complex L^2-space on $\mathbb{R}^d$, such that (B.9) holds for any $f, g \in \mathcal{L}^2(R^d)$, and that (B.10) holds for every $f \in \mathcal{L}^2(R^d)$ and every probability density h on $\mathbb{R}^d$.

Recall our notation

$$W^{1,2}(\mathbb{R}^d) = \Big\{ g \in \mathcal{L}^2(\mathbb{R}^d); \quad \nabla g \in \mathcal{L}^2(\mathbb{R}^d) \Big\}.$$

It is a well known fact that $W^{1,2}(\mathbb{R}^d)$ becomes a Hilbert space under the Sobolev norm

$$\sqrt{||g||_2^2 + ||\nabla g||_2^2}.$$

The isometric relation (B.11) can also be extended to $W^{1,2}(\mathbb{R}^d)$. More precisely, we have the following theorem.

Theorem B.2 *Given that $f \in \mathcal{L}^2(\mathbb{R}^d)$, $f \in W^{1,2}(\mathbb{R}^d)$ if and only if*

$$\int_{\mathbb{R}^d} |\lambda|^2 |\widehat{f}(\lambda)|^2 d\lambda < \infty.$$

Further, for any $f, g \in W^{1,2}(\mathbb{R}^d)$.

$$\int_{\mathbb{R}^d} \nabla f(x) \nabla g(x) dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\lambda|^2 \widehat{f}(\lambda) \overline{\widehat{g}(\lambda)} d\lambda.$$

Given a measure μ on $\mathbb{R}^d$ the *Fourier transform* $\widehat{\mu}(\cdot)$ of μ is defined as the function on $\mathbb{R}^d$ given as

$$\widehat{\mu}(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \cdot x} \mu(dx) \qquad \lambda \in \mathbb{R}^d \tag{B.12}$$

whenever the integral on the right hand side is well defined.

In particular, there is a one-to-one correspondence between μ and its Fourier transform $\widehat{\mu}$. In this book we are particularly interested in two opposite special cases — when μ is distributed on the lattice $\mathbb{Z}^d$ and when μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. It is a classic fact that μ is supported by the lattice $\mathbb{Z}^d$ if and only if $\widehat{\mu}(\lambda)$ is a *periodic function* with the period 2π:

$$\widehat{\mu}(\lambda + 2\pi y) = \widehat{\mu}(\lambda) \qquad \lambda \in \mathbb{R}^d, \quad y \in \mathbb{Z}^d.$$

In this case $\widetilde{\mu}(\lambda) = \widehat{f}(\lambda)$, where $f(x)$ is the density of μ with respect to the counting measure on $\mathbb{Z}^d$ and $\widehat{f}(\lambda)$ is given by (B.1).

As for the test of absolute continuity, we put the correspondent results into the following theorem known as *Planchrel-Parseval theorem.*

Theorem B.3 *A finite measure μ on $\mathbb{R}^d$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ if*

$$\int_{\mathbb{R}^d} |\widehat{\mu}(\lambda)|^2 d\lambda < \infty.$$

In this case we have the Parseval indentity, :

$$\int_{\mathbb{R}^d} f^2(x)dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\widehat{\mu}(\lambda)|^2 d\lambda$$

where $f(x)$ is the related density of μ.

Proof. For each $\epsilon > 0$, define

$$f_\epsilon(x) = \int_{\mathbb{R}^d} p_\epsilon(y-x)\mu(dy) \qquad x\in\mathbb{R}^d$$

where $p_\epsilon(x)$ is the normal density with mean zero and variance ϵ. We have

$$\widehat{f}_\epsilon(\lambda) = \exp\Big\{-\frac{\epsilon}{2}|\lambda|^2\Big\}\widehat{\mu}(\lambda).$$

In particular, $\widehat{f}_\epsilon(\cdot)$ $(\epsilon > 0)$ is a Cauchy sequence in $\mathcal{L}^2(\mathbb{R}^d)$ as $\epsilon \to 0^+$. So is the sequence $f_\epsilon(\cdot)$ $(\epsilon > 0)$ by Parseval identity. Let $f(\cdot)$ be the L^2-limit of $f_\epsilon(\cdot)$. One can clearly see that $f(x)$ is a density of μ and satisfies the Parsseval identity. $\square$

C Constant $\kappa(d,p)$ and related variations

Assume that $p(d-2)\le d$. The central topic is the best constant $\kappa(d,p)$ of the Gagliardo-Nirenberg inequality

$$||f||_{2p} \le C||\nabla f||_2^{\frac{d(p-1)}{2p}}||f||_2^{1-\frac{d(p-1)}{2p}} \qquad f\in W^{1,2}(\mathbb{R}^d) \tag{C.1}$$

where for any $r\ge 1$, $W^{1,r}(\mathbb{R}^d)$ denotes the *Sobolev space*

$$W^{1,r}(\mathbb{R}^d) = \big\{f\in\mathcal{L}^2(\mathbb{R}^d);\ \ \nabla f\in\mathcal{L}^r(\mathbb{R}^d)\big\}.$$

That is,

$$\kappa(d,p) = \inf\Big\{C>0;\ \ ||f||_{2p}\le C||\nabla f||_2^{\frac{d(p-1)}{2p}}||f||_2^{1-\frac{d(p-1)}{2p}} \\ \text{for all } f\in W^{1,2}(\mathbb{R}^d)\Big\}.$$

Gagliardo-Nirenberg inequality is closed related to *Sobolev inequality*, which claims (Theorem 2.4.1, p. 56, [169]) that for any $1 \le r < d$, there is a constant $K > 0$ such that

$$||f||_{r^*} \le K||\nabla f||_r \qquad f \in W^{1,r}(\mathbb{R}^d). \tag{C.2}$$

where $r* = dr(d-r)^{-1}$. In the special case $p(d-2) = d$, Gagliardo-Nirenberg inequality becomes the Sobolev inequality with $r = 2$. In the remaining case $p(d-2) < d$, Gagliardo-Nirenberg inequality can be derived from Sobolev inequality. Indeed, taking $r = d(p-1)p^{-1}$ in (C.2) we have

$$||f||_{d(p-1)} \le K||\nabla f||_{d(p-1)p^{-1}}.$$

Replacing f by $|f|^{\frac{2p}{d(p-1)}}$ gives

$$\begin{aligned}||f||_{2p} &\le \Big(\frac{2pK}{d(p-1)}\Big)^{\frac{d(p-1)}{2p}}\Big(\int_{\mathbb{R}^d}|\nabla f(x)|^{\frac{d(p-1)}{p}}|f(x)|^{\frac{2p-d(p-1)}{p}}dx\Big)^{1/2}\\ &\le \Big(\frac{2pK}{d(p-1)}\Big)^{\frac{d(p-1)}{2p}}||\nabla f||_2^{\frac{d(p-1)}{2p}}||f||_2^{1-\frac{d(p-1)}{2p}}\end{aligned}$$

where the second step follows from Cauchy-Schwartz inequality.

Other variations associated with $\kappa(d,p)$ are $M(\theta)$ and ρ defined below.

$$M(\theta) = \sup_{g\in\mathcal{F}_d}\Big\{\theta\Big(\int_{\mathbb{R}^d}|g(x)|^{2p}dx\Big)^{1/p} - \int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big\} \tag{C.3}$$

where

$$\mathcal{F}_d = \Big\{g \in \mathcal{L}^2(\mathbb{R}^d);\quad \int_{\mathbb{R}^d} g^2(x)dx = 1 \text{ and } \int_{\mathbb{R}^d}|\nabla g(x)|^2dx < \infty\Big\}$$

and

$$\rho = \sup\Big\{\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x-y)f(x)f(y)dxdy;\quad \int_{\mathbb{R}^d}|f(x)|^{\frac{2p}{2p-1}}dx = 1\Big\} \tag{C.4}$$

where

$$G(x) = \int_0^\infty e^{-t}p_t(x)dt \qquad x \in \mathbb{R}^d$$

is the Green's function of a d-dimensional Brownian motion.

Theorem C.1 *Assume that $p(d-2) < d$. For any $\theta > 0$,*

$$M(\theta) = \frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}}\kappa(d,p)^{\frac{4p}{2p-d(p-1)}}\theta^{\frac{2p}{2p-d(p-1)}}.$$

Proof. For any $f \in \mathcal{F}_d$,

$$\begin{aligned}
&\theta\bigg(\int_{\mathbb{R}^d} |f(x)|^{2p} dx\bigg)^{1/p} - \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx \\
&\le \theta\kappa(d,p)^2 ||\nabla f||_2^{\frac{d(p-1)}{d}} - \frac{1}{2}||\nabla f||_2^2 \\
&\le \sup_{\lambda\ge 0}\Big\{\theta\kappa(d,p)^2\lambda^{\frac{d(p-1)}{d}} - \frac{1}{2}\lambda^2\Big\} \\
&= \frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}}\kappa(d,p)^{\frac{4p}{2p-d(p-1)}}\theta^{\frac{2p}{2p-d(p-1)}}.
\end{aligned}$$

Taking supremum over $f \in \mathcal{F}_d$ on the left,

$$M(\theta) \le \frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}}\kappa(d,p)^{\frac{4p}{2p-d(p-1)}}\theta^{\frac{2p}{2p-d(p-1)}}.$$

On the other hand, for any $C < \kappa(d,p)$ there is a g such that

$$||g||_{2p} > C||\nabla g||_2^{\frac{d(p-1)}{2p}}||g||_2^{1-\frac{d(p-1)}{2p}}.$$

By homogeneity we may assume $||g||_2 = 1$. Given $\lambda > 0$, let $f(x) = \lambda^{d/2}g(\lambda x)$. Then $f \in \mathcal{F}_d$,

$$||\nabla f||_2 = \lambda||\nabla g||_2$$

and

$$||f||_{2p} = \lambda^{\frac{d(p-1)}{2p}}||g||_{2p} > C(\lambda||\nabla g||_2)^{\frac{d(p-1)}{2p}}.$$

Hence,

$$M(\theta) \ge \theta||f|||_{2p}^2 - \frac{1}{2}||\nabla f||_2^2 > \theta C^2(\lambda||\nabla g||_2)^{\frac{d(p-1)}{p}} - \frac{1}{2}(\lambda||\nabla g||_2)^2.$$

Since $\lambda > 0$ is arbitrary,

$$\begin{aligned}
M(\theta) &\ge \sup_{x>0}\Big\{\theta C^2 x^{\frac{d(p-1)}{p}} - \frac{1}{2}x^2\Big\} \\
&= \frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}}C^{\frac{4p}{2p-d(p-1)}}\theta^{\frac{2p}{2p-d(p-1)}}.
\end{aligned}$$

Letting $C \to \kappa(d,p)$ on the right hand side gives

$$\Psi(\theta) \ge \frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}}\kappa(d,p)^{\frac{4p}{2p-d(p-1)}}\theta^{\frac{2p}{2p-d(p-1)}}.$$

$\square$

Theorem C.2 *Assume that $p(d-2) < d$.*

$$\rho = \Big(\frac{2p - d(p-1)}{2p}\Big)^{\frac{2p-d(p-1)}{2p}} \Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p}} \kappa(d,p)^2.$$

Proof. By Theorem C.1, we need only to show that

$$M\Big(\frac{1}{\rho}\Big) = 1. \tag{C.5}$$

First, for any function f on $\mathbb{R}^d$ we introduce the notation

$$Gf(x) = \int_{\mathbb{R}^d} G(y-x) f(y)dx \qquad x \in \mathbb{R}^d$$

whenever the integral on the right hand side is well defined.

According to Theorem B in [157], the best constant $\kappa(d,p)$ can be attained at an infinitely smooth positive function. By examining the argument for Theorem Theorem C.1, one concludes that there there is a $f \in \mathcal{F}_d$ such that f is positive and infinitely smooth, and

$$M\Big(\frac{1}{\rho}\Big) = \frac{1}{\rho}\bigg(\int_{\mathbb{R}^d} f(x)^{2p}dx\bigg)^{1/p} - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx.$$

By *Lagrange multiplier*,

$$\frac{1}{\rho}\bigg(\int_{\mathbb{R}^d} f(x)^{2p}dx\bigg)^{\frac{p-1}{p}} f(x)^{2p-1} + \frac{1}{2}\triangle f(x) = M\Big(\frac{1}{\rho}\Big) f(x).$$

Write $\bar{f}(x) = f(x)/||f||_{2p}$. Then

$$\frac{1}{\rho}\bar{f}(x)^{2p-1} + \frac{1}{2}\triangle \bar{f}(x) = M\Big(\frac{1}{\rho}\Big)\bar{f}(x).$$

Multiplying $G\bar{f}^{2p-1}(x)$ to and integrating on both sides,

$$\frac{1}{\rho}\int_{\mathbb{R}^d} \bar{f}^{2p-1}(x) G\bar{f}^{2p-1}(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} \triangle \bar{f}(x) G\bar{f}^{2p-1}(x)dx$$
$$= M\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^d} \bar{f}(x) G\bar{f}^{2p-1}(x)dx.$$

So we have

$$1 + \frac{1}{2}\int_{\mathbb{R}^d} \triangle \bar{f}(x) G\bar{f}^{2p-1}(x)dx \geq M\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^d} \bar{f}(x) G\bar{f}^{2p-1}(x)dx.$$

Notice that

$$\frac{1}{2}\int_{\mathbb{R}^d}\triangle\bar f(x)G\bar f^{2p-1}(x)dx=\frac{1}{2}\int_{\mathbb{R}^d}\bar f(x)\triangle\circ G\bar f^{2p-1}(x)dx$$
$$=\int_{\mathbb{R}^d}\bar f(x)(G-I)\bar f^{2p-1}(x)dx=\int_{\mathbb{R}^d}\bar f(x)G\bar f^{2p-1}(x)dx-\int_{\mathbb{R}^d}f^{2p}(x)dx$$
$$=\int_{\mathbb{R}^d}\bar f(x)G\bar f^{2p-1}(x)dx-1,$$

where I is the identity operator on the function space, and the second equality follows from the *resolvent equation* (see (4.1.18) with $f\equiv 0$ and $\lambda=1$)

$$G=I+2^{-1}\triangle\circ G. \tag{C.6}$$

Hence,

$$\int_{\mathbb{R}^d}\bar f(x)G\bar f^{2p-1}(x)dx\ge M\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^d}\bar f(x)G\bar f^{2p-1}(x)dx$$

which leads to

$$M\Big(\frac{1}{\rho}\Big)\le 1.$$

On the other hand, let

$$c_0=\inf\Big\{\int_{\mathbb{R}^d}|f(x)|^2dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla f(x)|^2dx;\quad\int_{\mathbb{R}^d}|f(x)|^{2p}dx=1\Big\}.$$

We first claim that

$$\rho\le c_0^{-1}. \tag{C.7}$$

To this end, we show that for any function f on $\mathbb{R}^d$ with

$$\int_{\mathbb{R}^d}|f(x)|^{\frac{2p}{2p-1}}dx=1.$$

$$A\equiv\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(x-y)f(x)f(y)dxdy\le c_0^{-1}$$

Without loss of generality, we may assume that $A>0$. Notice that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(x-y)f(x)f(y)dxdy=\int_{\mathbb{R}^d}f(x)Gf(x)dx\le\Big\{\int_{\mathbb{R}^d}|Gf(x)|^{2p}dx\Big\}^{\frac{1}{2p}}.$$

Consequently,

$$\Big\{\int_{\mathbb{R}^d}|Gf(x)|^{2p}dx\Big\}^{\frac{1}{2p}}\ge A.$$

By the resolvent equation in (C.6),

$$\begin{aligned} A &= \langle Gf, f\rangle = \langle Gf,\ (I - 2^{-1}\Delta)\circ Gf\rangle \\ &= A^2\Big\langle \frac{Gf}{A},\ (I-2^{-1}\Delta)\frac{Gf}{A}\Big\rangle \ge A^2 c_0. \end{aligned}$$

This leads to (C.7).

By (C.7), for any $0<\epsilon<\rho$ there is a f on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |f(x)|^{2p}dx = 1$$

and that

$$\frac{1}{\rho-\epsilon} > \int_{\mathbb{R}^d}|f(x)|^2dx + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla f(x)|^2dx.$$

Let $g(x) = f(x)/||f||_2$. Then $g\in\mathcal{F}_d$ and

$$\begin{aligned} &\frac{1}{\rho-\epsilon}\Big\{\int_{\mathbb{R}^d}|g(x)|^{2p}dx\Big\}^{1/p} - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx \\ &\ge \Big\{||f||_2 + \frac{1}{2}||\nabla f||^2\Big\}||f||_2^{-2} - \frac{1}{2}||f||_2^{-2}||\nabla f||^2 \\ &= 1. \end{aligned}$$

Letting $\epsilon\to 0^+$ on the left-hand side leads to

$$M\Big(\frac{1}{\rho}\Big) \ge 1.$$

$\square$

By some obvious modification, Theorem C.1 and Theorem C.2 can be extended to the critical dimensions $p(d-2)=d$ (the case $d=p=3$ and the case $d=4$, $p=2$).

Theorem C.3 *In the critical dimensions defined by $p(d-2)=d$,*

$$M(\theta) = \begin{cases} 0 & \theta < 2\kappa(d,p)^2 \\ \infty & \theta > 2\kappa(d,p)^2, \end{cases}$$

$$\rho = 2\kappa(d,p)^2.$$

Finally, we compute the best constant of Gagliardo-Nirenberg inequality in the case $d=1$.

Theorem C.4 *Let $d = 1$. For any $p > 1$,*

$$M(\theta) = p^{-\frac{2p}{p+1}}\Big(\frac{\sqrt{2}}{(p-1)(p+1)}B\Big(\frac{1}{p-1}, \frac{1}{2}\Big)\Big)^{-\frac{2(p-1)}{p+1}}\theta^{\frac{2p}{p+1}} \qquad (\theta > 0) \quad \text{(C.8)}$$

$$\kappa(1,p) = p^{-1/2}\Big(\frac{\sqrt{2}}{(p-1)(p+1)}B\Big(\frac{1}{p-1}, \frac{1}{2}\Big)\Big)^{-\frac{p-1}{2p}} \qquad \text{(C.9)}$$

where $B(\cdot,\cdot)$ is beta function defined by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad a, b > 0.$$

Proof. By Theorem C.1, we need only to show (C.8) in the case $\theta = 1$. Our approach relies on Lagrange multiplier. By Theorem B in [157], there is an infinitely smooth, positive function $f(x)$ such that $f(-x) = f(x)$ for all $x \in \mathbb{R}$, that $f(x) \geq f(y)$ for $|x| \leq |y|$, and that

$$M(1) = \Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty}|f'(x)|^2dx.$$

By Lagrange multipliers, there is a $\lambda \in \mathbb{R}^d$ such that

$$\begin{aligned}&\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{\frac{p-1}{p}}\int_{-\infty}^{\infty} f^{2p-1}(x)g(x)dx - \int_{-\infty}^{\infty} f'(x)g'(x)dx\\ &= 2\lambda\int_{-\infty}^{\infty} f(x)g(x)dx\end{aligned}$$

for any $g \in W^{1,2}(\mathbb{R})$. Applying integration by parts,

$$\begin{aligned}&-2\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{\frac{p-1}{p}}\int_{-\infty}^{\infty} g'(x)\int_0^x f^{2p-1}(y)dydx - \int_{-\infty}^{\infty} f'(x)g'(x)dx\\ &= -2\lambda\int_{\infty}^{\infty} g'(x)\int_0^x f(y)dydx \qquad (x \in \mathbb{R}).\end{aligned}$$

Consequently,

$$\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{\frac{p-1}{p}}\int_0^x f^{2p-1}(y)dy + \frac{1}{2}f'(x) = \lambda\int_0^x f(y)dy.$$

Hence, $f'(0) = 0$ and

$$\Big(\int_{-\infty}^{\infty} f^{2p}(x)dx\Big)^{\frac{p-1}{p}} f^{2p-1}(x) + \frac{1}{2}f''(x)dx = \lambda f(x) \qquad x \in \mathbb{R}. \qquad \text{(C.10)}$$

Multiplying both side by $f(x)$ and integrating,

$$\lambda = \left(\int_{-\infty}^{\infty} f^{2p}(x)dx\right)^{1/p} - \frac{1}{2}\int_{-\infty}^{\infty} |f'(x)|^2 dx = M(1). \tag{C.11}$$

Multiplying by $f'(x)$ and then integrating the both sides of (C.10 we have, after simplification, that

$$\big(f'(x)\big)^2 = 2\left\{\lambda f^2(x) - \frac{1}{p}\left(\int_{-\infty}^{\infty} f^{2p}(x)dx\right)^{\frac{p-1}{p}} f^{2p}(x) + C\right\}$$

where, using the fact that $f'(0) = 0$,

$$C = \frac{1}{p}\left(\int_{-\infty}^{\infty} f^{2p}(x)dx\right)^{\frac{p-1}{p}} f^{2p}(0) - \lambda f^2(0). \tag{C.12}$$

Thus, by the fact that $f'(x) \le 0$ for $x \ge 0$,

$$dx = -\frac{1}{2}\left\{\lambda f^2(x) - \frac{1}{p}\left(\int_{-\infty}^{\infty} f^{2p}(x)dx\right)^{\frac{p-1}{p}} f^{2p}(x) + C\right\}^{-1/2} df(x) \tag{C.13}$$

for all $x \ge 0$. Consequently,

$$x = \frac{1}{2}\int_{f(x)}^{f(0)} \left\{\lambda y^2 - \frac{1}{p}\left(\int_{-\infty}^{\infty} f^{2p}(x)dx\right)^{\frac{p-1}{p}} y^{2p} + C\right\}^{-1/2} dy \qquad (x \ge 0).$$

This is impossible (can be seen when $x \to \infty$) unless $C = 0$. So by (C.12),

$$f(0) = \left(\int_{-\infty}^{\infty} f^{2p}(x)dx\right)^{\frac{1}{2p}} p^{\frac{1}{2p-2}} \lambda^{\frac{1}{2p-2}}. \tag{C.14}$$

Combining (C.13) and (C.14), by the fact that $f(\infty) = 0$

$$\begin{aligned}
&\int_{-\infty}^{\infty} f^{2p}(x)dx = 2\int_0^{\infty} f^{2p}(x)dx \\
&= \sqrt{2}\int_0^{f(0)} y^{2p}\left\{\lambda y^2 - \frac{1}{p}\left(\int_{-\infty}^{\infty} f^{2p}(x)dx\right)^{\frac{p-1}{p}} y^{2p} + C\right\}^{-1/2} dy \\
&= f^{2p}(0)\sqrt{\frac{2}{\lambda}}\int_0^1 \frac{u^{2p}du}{\sqrt{u^2 - u^{2p}}} \\
&= \sqrt{2} p^{\frac{p}{p-1}} \lambda^{\frac{p+1}{2(p-1)}} \left(\int_{-\infty}^{\infty} f^{2p}(x)dx\right) \int_0^1 \frac{u^{2p}du}{\sqrt{u^2 - u^{2p}}}.
\end{aligned}$$

Therefore

$$1 = \sqrt{2}p^{\frac{p}{p-1}}\lambda^{\frac{p+1}{2(p-1)}}\int_0^1 \frac{u^{2p}du}{\sqrt{u^2 - u^{2p}}}$$

Hence,

$$\lambda = p^{-\frac{2p}{p+1}}\left(\frac{\sqrt{2}}{(p-1)(p+1)}B\Big(\frac{1}{p-1}, \frac{1}{2}\Big)\right)^{-\frac{2(p-1)}{p+1}}.$$

Finally, the desired conclusion follows from (C.11). □

D Regularity of stochastic processes

Let $X = (X_t)_{t\in T}$ is a real stochastic process, where the index set T is compact under the metrics $d(s,t)$ $(s,t \in T)$. Our concern is the regularities of the process which includes continuity of its path and the integrability of its supremum norm.

A milestone paper in the study of regularity of the stochastic processes is due to Dudley ([59]). Dudley's idea can be roughly stated in the following way: One can install sample path continuity and boundedness for $X = (X_t)_{t\in T}$ by putting average Lipschitz condition (with respect to the distance $d(s,t)$) on the process and by controlling the size of the index set T. In Dudley's work, the size of T is measured by counting the small balls (under the metrics $d(s,t)$) necessary for covering T.

This powerful approach, known as the *entropy method* , has been developed into a variety of versions. The main part of this section is chosen from the book by Ledoux and Talagrand (Section 11.1, [114]) to fit the needs of our book.

A function Ψ: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called a *Yong function*, if it is increasing, convex and $\Psi(0) = 0$. Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a Yong function, the *Orlicz space* $\mathcal{L}_\Psi(\Omega, \mathcal{A}, \mathbb{P})$ is defined as the linear space of all random variables ξ on $(\Omega, \mathcal{A}, \mathbb{P})$ such that

$$||\xi||_\Psi \equiv \inf\Big\{c > 0; \quad \mathbb{E}\Psi(c^{-1}|\xi|) \le 1\Big\} < \infty.$$

It is a classic knowledge that $||\cdot||_\Psi$ define a norm on $\mathcal{L}_\Psi(\Omega, \mathcal{A}, \mathbb{P})$ known as *Orlicz norm* , and that $\mathcal{L}_\Psi(\Omega, \mathcal{A}, \mathbb{P})$ becomes a Banach space under the norm $||\cdot||_\Psi$.

For each $\epsilon > 0$, let $N(T, d, \epsilon)$ be the minimal number of open balls of radius ϵ which are necessary to cover T. A basic assumption is the following *entropy condition* :

$$\int_0^{D(T)} \Psi^{-1}\big(N(T,d,\epsilon)\big)d\epsilon < \infty \tag{D.1}$$

where $\Psi^{-1}(\cdot)$ is the inverse of $\Psi(\cdot)$, and

$$D(T) = \max\{d(s,t); \quad s,t \in T\}$$

is the diameter of T.

Theorem D.1 *Let $X = (X_t)_{t\in T}$ be a stochastic process in $\mathcal{L}(\Omega, \mathcal{A}, \mathbb{P})$ such that for any event $A \in \mathcal{A}$ and $s,t \in T$,*

$$\int_A |X_s - X_t| d\mathbb{P} \le d(s,t)\mathbb{P}(A)\Psi^{-1}\Big(\frac{1}{\mathbb{P}(A)}\Big). \tag{D.2}$$

Then, for any $A \in \mathcal{A}$,

$$\int_A \sup_{s,t\in T} |X_s - X_t| d\mathbb{P} \le 8d(s,t)\mathbb{P}(A)\int_0^{D(T)} \Psi^{-1}\Big(\frac{N(T,d,\epsilon)}{\mathbb{P}(A)}\Big)d\epsilon. \tag{D.3}$$

This result was established in Theorem 11.2, [114]. By slightly sharpen their estimate, Ledoux and Talagrand (Theorem 11.6, [114]) was able to derive a version of what is known as *Kolmogorov continuity theorem.*

Theorem D.2 *Let $X = (X_t)_{t\in T}$ be a stochastic process in $\mathcal{L}(\Omega, \mathcal{A}, \mathbb{P})$ and assume (D.1) and (D.2). Then $X = (X_t)_{t\in T}$ admits a modification $\widetilde{X} = (\widetilde{X}_t)_{t\in T}$ with all sample path bounded and (uniformly) continuous on (T,d). Moreover, for each $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\mathbb{E} \sup_{d(s,t)<\delta} |\widetilde{X}_s - \tilde{X}_t| < \epsilon. \tag{D.4}$$

For a stochastic process $X = (X_t)_{t\in T}$ with continuous sample path on T, X can be viewed as a random variable taking values in the Banach space $C(T)$, the space of continuous functions on T equipped with supremum norm. Given a family $\mathcal{X}$ of stochastic process $X = (X_t)_{t\in T}$ on T with continuous sample path on T, an important issue is the relative compactness of the family $\mathcal{X}$ with respect to the topology of weak convergence. For this purpose we recall some of the classic material on this subject.

Given a separable Banach space B, let $\mathcal{P}(B)$ be the space of all probability measures on B. Under the topology of weak convergence, $\mathcal{P}(B)$ becomes a separable complete metric space. A family $\{\mu_i\}_{i\in I}$ in $\mathcal{P}(B)$ is said to be *uniformly tight* , if for any $\epsilon > 0$ there is a compact set $K \subset B$ such that

$$\mu_i(K^c) < \epsilon \qquad i \in I. \tag{D.5}$$

The following well known *Prokhorov criterion* characterizes the relatively compact sets in $\mathcal{P}(B)$ in terms of the uniform tightness (see, e.g., Theorem 2.1 in [114]).

Theorem D.3 *The family $\mathcal{P}(B)$ is relatively compact in $\mathcal{P}(B)$ if and only if it is uniformly tight.*

To exam uniform tightness, one needs to look for the compact set $K \subset B$ such that (D.5) holds. When $B = C(T)$, the following classic *Arzelá-Ascoli theorem* provides criterion for relatively compact sets in $C(T)$.

Theorem D.4 *A family $\mathcal{F}$ in $C(T)$ is relatively compact if and only if for any (or for some) $t \in T$, $\{x(t);\ \ x \in \mathcal{F}$ is bounded and $\mathcal{F}$ is equicontinuous: For each $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\sup_{x\in\mathcal{F}} |x(s) - x(t)| < \epsilon$$

as soon as $d(s,t) < \delta$.

Given a family $\mathcal{X}$ of the stochastic processes on T such that every $X \in \mathcal{X}$ has continuous sample path and that (D.1) and (D.2) holds for every $X \in \mathcal{X}$, Ledoux and Talagrand's estimate allows the uniformity of (D.4) over $\mathcal{X}$, which, by continuity of X, states that for each $\epsilon > 0$ there is $\delta > 0$ such that

$$\sup_{X\in\mathcal{X}} \mathbb{E} \sup_{d(s,t)<\delta} |X_s - X_t| < \epsilon. \tag{D.6}$$

By this observation, Ledoux and Talagrand were able to establish the following theorem.

Theorem D.5 *Let $\mathcal{X}$ be a family of the continuous stochastic processes on T which are in $\mathcal{L}(\Omega, \mathcal{A}, \mathbb{P})$ and satisfy (D.1) and (D.2). Assume that for some $t_0 \in T$, the family $\{X_{t_0};\ \ X \in \mathcal{X}\}$ of the real random variables is uniformly tight in $\mathbb{R}$. Then $\mathcal{X}$ is uniformly tight in $C(T)$ and is therefore relatively compact in $\mathcal{P}\big(C(T)\big)$.*

Proof. Let $\epsilon > 0$ be fixed. By uniform tightness of $\{X_{t_0};\quad X \in \mathcal{X}\}$ there is a $M > 0$ such that

$$\mathbb{P}\{|X_{t_0}| > M\} < \frac{\epsilon}{2} \qquad X \in \mathcal{X}.$$

For each $k \geq 1$ by (D.6) and Chebyshev inequality there is a $\delta_k > 0$ such that

$$\mathbb{P}\Big\{\sup_{d(s,t)<\delta_k} |X_s - X_t| \geq \frac{1}{k}\Big\} \leq \frac{\epsilon}{2^{k+1}}.$$

By Arzelá-Ascoli theorem, the set

$$A = \bigcap_{k=1}^{\infty}\Big\{x \in C(T);\quad |x(t_0)| \leq M \text{ and } \sup_{d(s,t)<\delta_k} |x(s) - x(t)| \leq \frac{1}{k}\Big\}$$

is relative compact in $C(T)$. By our construction, $\mathbb{P}\{X \notin A\} < \epsilon$. Taking the compact set K as the closure of A in $C(T)$ leads to the uniform tightness of $\mathcal{X}$. □

To make them more practical to application, we now derive some consequences out of Theorem D.1, Theorem D.2 and Theorem D.5. The elegant argument we shall present comes from the book by Ledoux and Talagrand ([114]).

First, the condition (D.2) holds if

$$||X_s - X_t||_\Psi \leq d(s,t) \qquad s,t \in T. \tag{D.7}$$

Indeed, by Jensen's inequality

$$\begin{aligned}\int_A |X_s - X_t| d\mathbb{P} &= d(s,t)\mathbb{P}(A)\int_A \Psi^{-1}\circ\Psi\Big(\frac{|X_s - X_t|}{d(s,t)}\Big)\frac{d\mathbb{P}}{\mathbb{P}(A)} \\ &\leq d(s,t)\mathbb{P}(A)\Psi^{-1}\bigg(\frac{1}{\mathbb{P}(A)}\mathbb{E}\Psi\Big(\frac{|X_s - X_t|}{d(s,t)}\Big)\bigg) \\ &\leq d(s,t)\mathbb{P}(A)\Psi^{-1}\Big(\frac{1}{\mathbb{P}(A)}\Big).\end{aligned}$$

Second, if we assume that[7]

$$\Psi^{-1}(xy) \leq C\Psi^{-1}(x)\Psi^{-1}(y) \qquad x,y \geq 1 \tag{D.8}$$

[7]Roughly speaking, this means that the increasing rate of $\Psi(x)$ is no less than a linear function, an assumption that agrees at large with the monotonicity and convexity of Ψ.

then (D.3) leads to

$$\mathbb{P}\Big\{\sup_{s,t\in T}|X_s - X_t| \geq u\Big\} \leq \left(\Psi\Big(\frac{u}{8EC}\Big)\right)^{-1} \qquad u > 0 \tag{D.9}$$

where

$$E = \int_0^{D(T)} \Psi^{-1}\big(N(T,d,\epsilon)\big)d\epsilon.$$

Indeed, by (D.3) and (D.8)

$$\int_A |X_s - X_t| d\mathbb{P} \leq 8CE\mathbb{P}(A)\Psi^{-1}\Big(\frac{1}{\mathbb{P}(A)}\Big).$$

Take $A = \Big\{\sup_{s,t\in T}|X_s - X_t| \geq u\Big\}$. By Chebyshev inequality,

$$\mathbb{P}(A) \leq \frac{1}{u} 8CE\mathbb{P}(A)\Psi^{-1}\Big(\frac{1}{\mathbb{P}(A)}\Big)$$

which proves to (D.9).

For the applications to this book, we take T as a compact domain in $\mathbb{R}^N$ with $N \geq 1$, and $d(s,t) = c|s-t|^\delta$, where $0 < \delta \leq 1$ and $|\cdot|$ is the Euclidean norm. One can see that in this case

$$N(T,d,\epsilon) = O\Big(\epsilon^{-N\delta^{-1}}\Big) \qquad (\epsilon \to 0^+).$$

Consequently, (D.1) holds if

$$\int_0^1 \Psi^{-1}\big(\epsilon^{-N\delta^{-1}}\big)d\epsilon < \infty. \tag{D.10}$$

As for the Yong function Ψ we only consider two special cases $\Psi(\theta) = \exp\{\theta^r\} - 1$ for some $r > 0$ and $\Psi(\theta) = \theta^m$ for some $m > N$.

In the first case, $\Psi(\theta)$ is not a Yong function when $r < 1$, as the convexity fails for small θ. To fix this problem, one can easily construct a Yong function $\widetilde{\Psi}$ satisfying (D.8) such that $\Psi(\theta) = \widetilde{\Psi}(\theta)$ for large θ and apply what has been achieved to $\widetilde{\Psi}$. Notice that in this case (D.10) automatically holds. Here is what we get.

Theorem D.6 *Let $X = (X_t)_{t\in T}$ be a stochastic process in $\mathcal{L}(\Omega, \mathcal{A}, \mathbb{P})$, where T is a compact domain in $\mathbb{R}^N$. Let $0 < \delta \leq 1$ and $r > 0$ be fixed and assume that*

$$\sup_{\substack{s,t\in T\\ s\neq t}} \mathbb{E}\exp\left\{c\frac{|X_s - X_t|^r}{|s-t|^\delta}\right\} < \infty \tag{D.11}$$

for some $c > 0$. Then $X = (X_t)_{t\in T}$ admits a modification $\widetilde{X} = (\widetilde{X}_t)_{t\in T}$ with all sample path bounded and (uniformly) continuous on (T, d).

In addition, there is $c' > 0$ such that

$$\mathbb{E}\exp\left\{c' \sup_{s,t\in T} |X_s - X_t|^r\right\} < \infty. \tag{D.12}$$

Given a family $\mathcal{X}$ of stochastic processes on T such that (D.11) holds uniformly for all $X \in \mathcal{X}$,

$$\sup_{X\in\mathcal{X}} \mathbb{E}\exp\left\{c'' \sup_{s,t\in T} |X_s - X_t|^r\right\} < \infty \tag{D.13}$$

for some $c'' > 0$.

Proof. All we need to say is that (D.9) leads to (D.12) (and (D.13) in the later case) for our choice of Ψ. □

As for the second choice $\Psi(\theta) = \theta^m$, notice that (D.10) holds if $m\delta > N$. Therefore, we have the following version of Kolmogorov continuity theorem.

Theorem D.7 *Let $X = (X_t)_{t\in T}$ be a stochastic process in $\mathcal{L}(\Omega, \mathcal{A}, \mathbb{P})$, where T is a compact domain in $\mathbb{R}^N$. Assume that there are constants $0 < \delta \le 1$ and $m > N$ such that $m\delta > N$ and that*

$$\mathbb{E}|X_s - X_t|^m \le M|s - t|^{m\delta} \qquad s, t \in T \tag{D.14}$$

for some constant $M > 0$. Then $X = (X_t)_{t\in T}$ admits a modification $\widetilde{X} = (\widetilde{X}_t)_{t\in T}$ with all sample path bounded and (uniformly) continuous on (T, d).

Further, a family $\mathcal{X}$ of $C(T)$-valued random variables is uniformly tight in $C(T)$ and is therefore relatively compact in $\mathcal{P}\big(C(T)\big)$, if for some $t \in T$, the family $\{X_{t_0};\ \ X \in \mathcal{X}\}$ of the real random variables is uniformly tight in $\mathbb{R}$, and if (D.14) holds uniformly for all $X \in \mathcal{X}$.

E Self-adjoint operators

Given a $N \times N$ symmetric matrix with real elements, there is a $N \times N$ orthogonal matrix T such that $A = TDT^*$, where D is the diagonal matrix with the eigenvalues $\lambda_1 \le \cdots \le \lambda_m$ of A as its diagonal elements (in the increasing order), and T^* is the transpose of T. This representation can be rewritten as

$$A = \sum_{k=1}^{N} \lambda_k \big(E(k) - E(k-1)\big) \tag{E.1}$$

where $E(0)$ is $N \times N$ 0-matrix and $E(k) == TD_kT^*$, where D_k is the diagonal matrix whose first k diagonal elements are 1's and remaining are 0's. For each $0 \le k \le N$, $E(k)$ projects $\mathbb{R}^N$ into a k-dimensional subspace of $\mathbb{R}^N$ and for $1 \le j \le k \le N$, $E(j)E(k) = E(k)E(j) = E(j)$. For any $x \in \mathbb{R}^N$, $\mu_x(k) = \langle x, E(k) - E(k-1)x\rangle \ge 0$ $(1 \le k \le N)$ and

$$\sum_{k=1}^{N} \mu_x(k) = |x|^2. \tag{E.2}$$

The representation in (E.1) is the most classic example of *spectral decomposition*. For any integer $m \ge 0$, A^m is symmetric and the equality $A^m = TD^mT^*$ holds for the same T and D. Therefore,

$$A^m = \sum_{k=1}^{N} \lambda_k^m \big(E(k) - E(k-1)\big). \tag{E.3}$$

Given a real polynomial

$$\xi(\lambda) = \sum_{i=0}^{m} c_i \lambda^i.$$

One can define the matrix $\xi(A)$ in the following natural way:

$$\xi(A) = \sum_{i=0}^{m} c_i A^i.$$

By linearty and (E.3)

$$\xi(A) = \sum_{k=1}^{N} \xi(\lambda_k)\big(E(k) - E(k-1)\big). \tag{E.4}$$

For a more general function $\xi(\lambda)$, one can define the matrix $\xi(A)$ by (E.4).

This elementary observation can be extended to a class of linear operators known as self-adjoint operators on a Hilbert space and the topic is called *spectral representation*. The case of bounded self-adjoint operators has become standard material for a graduate course on functional analysis. In the following we consider the general case in which A is not necessarily bounded. The most of the material on general theory of self-adjoint operators in this section is taken from the book by Yosida ([168]).

In the following discussion, H is a separable real Hilbert space. A linear operator A on H is said to be *densely defined* if its domain, denoted by $D(A)$, is dense in H. The *adjoint operator* is the operator A^*: $y \mapsto y^*$ defined by the relation

$$\langle Ax, y\rangle = \langle x, y^*\rangle \qquad x \in D(A).$$

A densely defined operator is said to be *symmetric*, if

$$\langle Ax, y\rangle = \langle x, Ay\rangle \qquad x, y \in D(A).$$

Clearly, $D(A) \subset D(A^*)$ and $Ay = A^*y$ for each $y \in D(A)$ when A is symmetric. A symmetric operator is said to be *self-adjoint* , if $D(A) = D(A^*)$. In other words, A and A^* are the same operator when A is self-adjoint. In the special case when A is bounded, there is no difference between being symmetric and being self-adjoint. This is no longer the case when it comes to unbounded operator. See Example 4, Section VII in [168] for an example of non-self-adjoint but symmetric operators. In many aspects, it is the self-adjoint operators who appear as a natural extension of symmetric matrices.

A *projection operator* P is a bounded self-adjoint operator on H with $P^2 = P$. It is easy to see that $||P|| \le 1$ (the equality holds when P is not zero operator) and that P is symmetric. Consequently, a projection operator is self-adjoint.

A family $\{E(\lambda);\ \ -\infty < \lambda < \infty\}$ of projection operators on H is called a *resolution of the identity*, if

(1) $E(\lambda) \circ E(\gamma) = E(\min\{\lambda, \gamma\})$;

(2). $E(-\infty)$ is zero operator, $E(\infty)$ is identity operator, and $E(\lambda + 0) = E(\lambda)$ for every $\lambda \in \mathbb{R}$, where $E(-\infty)$, $E(\infty)$ and $E(\lambda + 0)$ are the linear operators defined as

$$E(\pm\infty)(x) = \lim_{\lambda \to \pm\infty} E(\lambda)(x), \qquad E(\lambda + 0)(x) = \lim_{\gamma \to \lambda^+} E(\gamma)(x) \qquad \forall x \in H.$$

By definition, for any $x \in H$, the function

$$F(\lambda) = \langle E(\lambda)(x), x\rangle = ||E(\lambda)(x)||^2$$

is a distribution function on $\mathbb{R}$. We write μ_x for the measure generated by F and call μ_x *spectral measure*. Clearly, μ_x is finite and, correspondent to (E.2),

$$\mu_x(\mathbb{R}) = ||x||^2 \qquad x \in H. \tag{E.5}$$

There is an obvious similarity between the definitions of resolution of the identity and probability distribution function. Recall that a probability distribution function is defined by four properties: monotonicity, right continuity, vanishing at $-\infty$, and increasing to 1 at ∞. If we define the order $P_1 \le P_2$ between two projection operators P_1 and P_2 on H by the relation $P_1(H) \subset P_2(H)$, then resolution of the identity is defined in a way same as how probability distribution function is defined.

Given a suitable function $\xi(\lambda)$ on $\mathbb{R}$ and a distribution function $F(\lambda)$, the integral

$$\int_{-\infty}^{\infty} \xi(\lambda) F(d\lambda)$$

is extensively studied in every first-year graduate book in probability. Similarly, we intend to define the linear (possibly unbounded) operator in the form

$$\int_{-\infty}^{\infty} \xi(\lambda) E(d\lambda)$$

and call it *spectral integral*.

In the special case when ξ is a step function supported on a finite interval $[a, b]$ and is piece-wisely defined with respect to a partition $a = \lambda_0 < \lambda_1 \cdots < \lambda_n = b$, we define

$$\int_{-\infty}^{\infty} \xi(\lambda) E(d\lambda) = \sum_{k=1}^{n} c_k \big(E(\lambda_k) - E(\lambda_{k-1}) \big)$$

where c_k is the value of the function $\xi(\cdot)$ on the k-the sub-interval $(\lambda_{k-1}, \lambda_k]$.

By this definition one can immediately see that for each $x \in H$,

$$\left\| \left(\int_{-\infty}^{\infty} \xi(\lambda) E(d\lambda) \right)(x) \right\|^2 = \int_{-\infty}^{\infty} |\xi(\lambda)|^2 \mu_x(d\lambda) \tag{E.6}$$

and

$$\left\langle \left(\int_{-\infty}^{\infty} \xi(\lambda) E(d\lambda) \right)(x),\ x \right\rangle = \int_{-\infty}^{\infty} \xi(\lambda) \mu_x(d\lambda). \tag{E.7}$$

By a standard argument of approximation, the definition is extended to a more general class of $\xi(\cdot)$. More precisely, for any Borel-measurable $\xi(\lambda)$ on $\mathbb{R}$, the linear operator

$$\left(\int_{-\infty}^{\infty} \xi(\lambda) \mathbb{E}(d\lambda) \right)(x) \qquad x \in \mathcal{D}_\xi \tag{E.8}$$

is defined with the domain $\mathcal{D}_\xi \subset H$ is given by

$$\mathcal{D}_\xi = \left\{ x \in H; \quad \int_{-\infty}^{\infty} |\xi(\lambda)|^2 \mu_x(d\lambda) < \infty \right\}.$$

Further, (E.6) and (E.7) hold for every $x \in \mathcal{D}_\xi$. By the fact that μ_x is a finite measure, by Cauchy-Schwartz inequality

$$\int_{-\infty}^{\infty} |\xi(\lambda)| \mu_x(d\lambda) < \infty$$

whenever $x \in \mathcal{D}_\xi$. Consequently, the right hand side of (E.7) is well defined.

A striking fact is that the linear operator in (E.8) is self-adjoint (Theorem 2, Section XI.5, [168]).

Correspondent to (E.1), the following theorem (Theorem 1, Section XI.6, [168]) claims that the linear operators in the form of (E.8) are the only kind of self-adjoint operators.

Theorem E.1 *Given a self-adjoint operator A, there is an unique resolution of identity $\{E(\lambda); \quad -\infty < \lambda < \infty\}$, such that*

$$A = \int_{-\infty}^{\infty} \lambda E(d\lambda), \tag{E.9}$$

$$\mathcal{D}(A) = \left\{ x \in H; \quad \int_{-\infty}^{\infty} |\lambda|^2 \mu_x(d\lambda) < \infty \right\}. \tag{E.10}$$

The discussion in the rest of the section is based on the representation (E.9).

The self-adjoint operator A is said to be *upper semi-bounded*, (or *lower semi-bounded*), if

$$\lambda_0 \equiv \sup_{\substack{x \in \mathcal{D}(A) \\ ||x||=1}} \langle x, Ax \rangle < \infty \qquad \Big(\text{or } \lambda_0' \equiv \inf_{\substack{x \in \mathcal{D}(A) \\ ||x||=1}} \langle x, Ax \rangle > -\infty\Big)$$

It is easy to see that a bounded self-adjoint operator A is upper and lower semi bounded and

$$||A|| = \sup_{||x||=1} \Big| \langle x, Ax \rangle \Big|.$$

The following theorem is used multiple times in this book.

Theorem E.2 *Let the self-adjoint operator A be given in (E.9) and let $x \in H$ be arbitrary. When A is upper-semi bounded the smallest supporting set of the spectral measure μ_x is bounded from above by λ_0. When A is lower-semi bounded the smallest supporting set of μ_x is bounded from below by λ_0'.*

Proof. Due to similarity we only consider the case when A is upper-semi bounded. We use argument by contradiction. We show that if there were $x_0 \in H$ and $\lambda_1 > \lambda_0$ such that $\mu_{x_0}(\lambda_1, \infty) > 0$, there would be $x^* \in \mathcal{D}(A)$ such that $||x^*|| = 1$ and that $\langle x^*, Ax^* \rangle > \lambda_0$.

Indeed, one can find $\lambda_2 > \lambda_1$ such that

$$\mu_{x_0}(\lambda_1, \lambda_2) > 0.$$

Let $x_1 = E(\lambda_2)x_0 - \mathbb{E}(\lambda_1)x_0$. By the definition of resolution of identity,

$$\begin{aligned} ||x_1||^2 &= ||E(\lambda_2)x_0||^2 + ||E(\lambda_1)x_0||^2 - 2\langle E(\lambda_1)x_0, E(\lambda_2)x_0 \rangle \\ &= ||E(\lambda_2)x_0||^2 + ||E(\lambda_1)x_0||^2 - 2||E(\lambda_1)x_0||^2 \\ &= \langle x_0, E(\lambda_2)x_0 \rangle - \langle x_0, E(\lambda_1)x_0 \rangle = \mu_{x_0}(\lambda_1, \lambda_2] > 0. \end{aligned} \tag{E.11}$$

In particular, $x_1 \neq 0$.

In addition, it is easy to check that $\langle x_1, E_(\lambda)x_1 \rangle = ||x_1||^2$ when $\lambda \geq \lambda_2$, and $\langle x_1, E(\lambda)x_1 \rangle = 0$ when $\lambda < \lambda_1$. Since $\mu_{x_1}(\mathbb{R}) = ||x_1||^2$, μ_{x_1} is supported by $(\lambda_1, \lambda_2]$. Thus,

$$\int_{-\infty}^{\infty} |\lambda|^2 \mu_{x_1}(d\lambda) = \int_{(\lambda_1, \lambda_2]} |\lambda|^2 \mu_{x_1}(d\lambda) < \infty.$$

Consequently, $x_1 \in \mathcal{D}(A)$.

For the same reason,

$$\langle x_1, Ax_1 \rangle = \int_{-\infty}^{\infty} \lambda \mu_{x_1}(d\lambda) = \int_{(\lambda_1, \lambda_2]} \lambda \mu_{x_1}(d\lambda) \geq \lambda_1 \mu_{x_0}(\lambda_1, \lambda_2] = \lambda_1 ||x_1||_2^2$$

where the last step follows from (E.11).

Finally, letting $x^* = ||x_1||^{-1} x_1$ completes the proof. □

In the light casted by the representation in (E.4), given a self-adjoint operator A in the form of (E.9) *the function* of A is defined as the self-adjoint operator

$$\xi(A) = \int_{-\infty}^{\infty} \xi(\lambda) E(d\lambda) \tag{E.12}$$

where $\xi(\cdot)$ is a Borel function on $\mathbb{R}$.

The following are some immediate observations. The domain of $\xi(A)$ is determined as

$$\mathcal{D}\big(\xi(A)\big) = \left\{x \in H;\ \ \int_{-\infty}^{\infty} |\xi(\lambda)|^2 \mu_x(d\lambda) < \infty\right\}. \tag{E.13}$$

By (E.6) and (E.7), for any $x \in \mathcal{D}\big(\xi(A)\big)$

$$\big|\big|\xi(A)x\big|\big|^2 = \int_{-\infty}^{\infty} |\xi(\lambda)|^2 \mu_x(d\lambda), \tag{E.14}$$

$$\langle \xi(A)x,\ x\rangle = \int_{-\infty}^{\infty} \xi(\lambda)\mu_x(d\lambda). \tag{E.15}$$

By the definition given in (E.12), some standard properties enjoined by integrals, such as the linearty, hold naturally (see Theorem 3, Section XI.12, [168]). In addition to that, there are some truly remarkable properties in operational calculus, such as the one given in the following theorem (Part (iv) of Theorem 3, Section XI.12, [168]).

Theorem E.3 *Let A be a self-adjoint operator. For two Borel functions $\xi(\lambda)$ and $\eta(\lambda)$ on $\mathbb{R}$, and for $x \in \mathcal{D}\big(\xi(A)\big)$, $\xi(A)x \in \mathcal{D}\big(\eta(A)\big)$ if and only if $x \in \mathcal{D}\big((\eta \cdot \xi)(A)\big)$. In this case*

$$\big(\eta(A) \circ \xi(A)\big)(x) = (\eta \cdot \xi)(A)(x).$$

In some special cases, Theorem E.3 connects different approaches of creating the functions of self-adjoint operators. To show our point, we assume that the self-adjoint operator A is bounded. Given a integer $m \geq 2$, on the one hand, the power A^m can be defined by the spectral integral

$$\int_{-\infty}^{\infty} \lambda^m E(d\lambda).$$

On the other hand, A^m can be defined as the composition

$$A^m = \overbrace{A \circ \cdots \circ A}^{m}. \tag{E.16}$$

A natural question is whether these two approaches agree. Observe that the domain of a bounded operator is the whole space H, we have the following direct corollary from Theorem E.3.

Corollary E.4 *Given the bounded self-adjoint operator A in the form (E.9),*

$$A^m = \int_{-\infty}^{\infty} \lambda^m E(d\lambda) \qquad m = 1, 2, \cdots$$

where A^m is defined in (E.16).

In general, identifying the operator $\xi(A)$ given in (E.12) with the ones defined in other ways (such as defining $\xi(A)$ by Taylor expansion or by polynomial approximation) appears to be highly non-trival. Such problems have to be treated in case by case basis. By comparing (E.10) and (E.13), for example, it is not hard to image (see Section 4.1 for example $\xi(\lambda) = e^{t\lambda}$) that some function $\xi(A)$ of an unbounded self-adjoint operator A may have a domain $\mathcal{D}\big(\xi(A)\big)$ genuinely larger than $\mathcal{D}(A)$. On the other hand, the domain of any polynomial of A of degree at least 1 is subset to $\mathcal{D}(A)$. Consequently, approximation of $\xi(A)$ by polynomials of A leads to a different operator in this case. A big advantage of defining $\xi(A)$ by (E.12) is that it largely reduce the investigation of the operator $\xi(A)$ to the ordinary integration problems through the relations (E.4), (E.13) and (E.14).

Bibliography

[1] Asselah, A. and Castell, F. (2006) Self-intersection local times for random walk, and random walk in random scenery in dimension $d \geq 5$. *Probab. Theor. Rel. Fields* (to appear).

[2] Asselah, A. (2006). Large deviation estimates for self-intersection local times for simple random walk in $\mathbb{Z}^3$. (preprint).

[3] Asselah, A. (2008). On large deviation and self-Intersection local times in dimension five or more. (preprint).

[4] Bass, R. F. and Chen, X. (2004). Self-intersection local time: critical exponent and laws of the iterated logarithm. *Ann. Probab.* **32** 3221-3247.

[5] Bass, R. F., Chen, X. and Rosen, J. (2005). Large deviations for renormalized self-intersection local times of stable processes. *Ann. Probab.* **33** 984-1013.

[6] Bass, R. F., Chen, X. and Rosen, J. (2006). Moderate deviations and laws of the iterated logarithm for the renormalized self-intersection local times of planar random walks. *Electron. J. Probab.* **11** 933-1030.

[7] Bass, R. F., Chen, X. and Rosen, J. (2007). Moderate deviations for the range of planar random walks. *Memoirs of AMS.* (to appear).

[8] Bass, R. F., Chen, X. and Rosen, J. (2008). Large deviations for Riesz potentials of Additive Processess. *Annales de l'Institut Henri Poincare* (to appear).

[9] Bass, R. F. and Kumagai, T. (2002). Laws of the iterated logarithm for the range of random walks in two and three dimensions. *Ann. Probab.* **30** 1369-1396.

[10] van den Berg, M., Bolthausen, E. and den Hollander, F. (2001). Moderate deviations for the volume of Wiener sausage. *Ann. Math.* **153** 335-406.

[11] van den Berg, M., Bolthausen, E. and den Hollander, F. (2004). On the volume of the intersection of two Wiener sausages. *Ann. Math.* **159** 741-785.

[12] Biscup, M. and König, W. (2001). Long-time tails in the parabolic Anderson model with bounded potential. *Ann. Probab.* **29** 636-682.

[13] Bolthausen, E. (1989). A central limit theorem for two-dimensional random walks in random sceneries. *Ann. Probab.* **17** 108-115.

[14] Bolthausen, E. (1990). On the volume of Wiener sausage. *Ann. Probab.* **18** 1576-1582.

[15] Bolthausen, E. (1999). Large deviations and interacting random walks. *Lecture Notes in Math.* **1781** 1-124.

[16] Breiman, L. (1968). *Probability.* Addison-Wesley Massachusetts.

[17] Brydges, D. S. and Slade, G. (1995). The diffusive phase of a model of self-interracting walks.

[18] Bucklew, J. A. (1990). *Large Deviations Techniques in Decision, Simulation, and Estimation.* Wiley, New York.

[19] Buffet, E. and Pulé, J. V. (1997) A model of continuous polymers with random charges. *J. Math. Phys.* **38** 5143–5152.

[20] Calais, J. Y. and Yor, M. (1987). Renormalization et convergence en loi pour certaines intégrales multiples associées au mouvement brownien dans $\mathbb{R}^d$. *Séminaire de Probabilités XXI. Lecture Notes in Math.* **1247** 375–403. Springer, Berlin.

[21] Carlen, E. A. and Loss, M. (1993). Sharp constant in Nash's inequality. *Internat. Math. Res. Notices* **1993** 213-215.

[22] Carmona, R. and Molchanov, S. A. (1994). *Parabolic Anderson Problem and Intermittency. Amer. Math. Soc.,* Providence, RI.

[23] Chen, X. (1999). How often does a Harris recurrent Markov chain recur? *Ann. Probab.* **3** 1324-1346.

[24] Chen, X. (2004). Exponential asymptotics and law of the iterated logarithm for intersection local times of random walks. *Ann. Probab.* **32** 3248-3300.

[25] Chen, X. (2005). Moderate deviations and law of the iterated logarithm for intersections of the range of random walks. *Ann. Probab.* **33** 1014–1059.

[26] Chen, X. (2006). Self-intersection local times of additive processes: large deviation and law of the iterated logarithm. *Stoc. Proc. Appl.* **116** 1236-1253.

[27] Chen, X. (2006). Moderate and small deviations for the ranges of one-dimensional random walks. *J. Theor. Probab.* **19** 721-739.

[28] Chen, X. (2007). Large deviations and laws of the iterated logarithm for the local times of additive stable processes. *Ann. Probab.* **35** 602-648.

[29] Chen, X. (2007). Moderate deviations and laws of the iterated logarithm for the local times of additive Levy processes and additive random walks. *Ann. Probab.* **35** 954-1006.

[30] Chen, X. (2007c). Limit laws for the energy of a charged polymer. *Annales de l'Institut Henri Poincare* (to appear).

[31] Chen, X. and Li, W. (2004).Large and moderate deviations for intersection local times. *Probab. Theor. Rel. Fields.* **128** 213-254.

[32] Chen, X., Li, W. and Rosen, J. (2005). Large deviations for local times of stable processes and stable random walks in 1 dimension. *Electron. J. Probab.* **10** 577-608.

[33] Chen, X. and Mörters, P. (2007). Upper tails for intersections of random walks in supercritical dimensions. (preprint).

[34] Chen, X., and Rosen, J. (2005). Exponential asymptotics for intersection local times of stable processes and random walks. *Ann. Inst. H. Poincaré* **41** 901-928.

[35] Clifford, W. K. (1882). On Bessel's functions, *Mathematical Papars* 246-349.

[36] Cordero-Erausquin, D., Nazaret, B. and Villani, C. (2004). A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. *Adv. Math.* **182** 307-332.

[37] Cramér, H. (1938). Sur un nouveau théorème-limite de la théorie des probabilités. In *Actualités Scientifiques et Industrielles, number 736 in Colleque Consacré à la Théorie des Probabilités, page 5-23* Hermann, Paris.

[38] Cranston, M., Mountford, T. S. and Shiga, T. (2005). Lyapunov exponent for the parabolic Anderson model with Lévy noise. *Probab. Theory Related Fields* **132** 321–355.

[39] Csáki, E., König, W. and Shi Z. (1999). An embedding for Kesten-Spitzer random walk in random scenery. *Stochastic Process. Appl.* **82** 283-292.

[40] Csörgö, M., Shi, Z. and Yor, M. (1999). Some asymptotic properties of the local time of the uniform empirical process. *Bernoulli* **5** 1035-1058.

[41] de Acosta, A. (1985). Upper bounds for large deviations of dependent random vectors. *Z. Wahrsch. Verw. Gebiete* **69** 551–565.

[42] Del Pino, M. and Dolbeault, J. (2002). Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl.* **81** 847-875.

[43] Del Pino, M. and Dolbeault, J. (2003). The optimal Euclidean L^p-Sobolev logarithmic inequality. *J. Funct. Anal.* **197** 151-161.

[44] Dembo, A. and Zeitouni, O. (1998). *Large Deviations Techniques and Applications. (2nd ed.)* Springer, New York.

[45] Dembo, A., Peres, Y., Rosen, J. and Zeitouni, O. (2001). Thick Points for Planar Brownian Motion and the Erdös-Taylor Conjecture on Random Walk. *Acta Math.* **186** 239-270.

[46] Dembo, A., Peres, Y. and Rosen, J. (2007). How large a disc is covered by a random walk in n steps? *Ann. Probab.* **35** 577-601.

[47] den Hollander, F. (2000). *Large Deviations.* Amer. Math. Soc., Providence, RI.

[48] Derrida, B., Griffiths, R. B. and Higgs, R. G. (1992). A model of directed walks with random self interactions. *Europhys. Lett.* **18** 361-366.

[49] Derrida, B.and Higgs, P. G. (1994) Low-temperature properties of directed walks with random self-interactions. *J. Phys. A* **27** 5485-5493.

[50] Deuschel, J. D. and Stroock, D. W. (1989). *Large Deviations* Academic Press, Boston.

[51] Donoghue, W. (1969). *Distributions and Fourier transforms.* Academic Press, New York.

[52] Donsker, M. D. and Varadhan, S. R. S. (1974). Asymptotic evaluation of certain Wiener integrals for large time. *Functional Integration and Its Applications: Proceedings of the International Conference at London.* p15-33.

[53] Donsker, M. D. and Varadhan, S. R. S. (1975). Asymptotic evaluation of certain Markov process expectations for large time. I. *Comm. Pure Appl. Math.* **28** 1-47, 279-301.

[54] Donsker, M. D.and Varadhan, S. R. S. (1975). Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.* **28** 525–565

[55] Donsker, M. D. and Varadhan, S. R. S. (1976). Asymptotic evaluation of certain Markov process expectations for large time. III. *Comm. Pure Appl. Math.* **29** 389-461.

[56] Donsker, M. D. and Varadhan, S. R. S. (1979). On the number of distinct sites visited by a random walk. *Comm. Pure Appl. Math.* **32** 721-747.

[57] Donsker, M. D. and Varadhan, S. R. S. (1981). The polaron problem and large deviations. *New stochastic methods in physics. Phys. Rep.* **77** 235-237.

[58] Donsker, M. D.and Varadhan, S. R. S. (1983). Asymptotics for the polaron. *Comm. Pure Appl. Math.* **36** 505-528.

[59] Dudley, R. (1967). The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *J. Funct. Anal.* **1** 290-330.

[60] Duplanier, B. Private communication.

[61] Duplanier, B. (1986). Exact critical exponents for two-dimensional dense polymers. *J. Phys. A* **19** L1009-L1014.

[62] Duplanier, B. (2003).Higher conformal multifractality. *J. Statist. Phys.* **110** 691–738.

[63] Dvoretzky, A. and Erdös, P. (1951). Some problems on random walk in space. *Proc. Secon Berkeley Symposium on Math. Statistics and Probability* 353-367. Berkeley, CA: University of California Press.

[64] Dvoretzky, A., Erdös, P. and Kakutani, S. (1950). Double points of paths of Brownian motions in n-space. *Acta Sci. Math. Szeged* **12**, Leopoldo, Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 75-81.

[65] Dvoretzky, A., Erdös, P. and Kakutani, S. (1954). Multiple points of paths of Brownian motion in the plane. *Bull. Res. Council Israel.* **3**, 364-371

[66] Dynkin, E. B. (1981). Additive functionals of several time-reversible Markov processes. *J. Funct. Anal.* **42** 64-101.

[67] Dynkin, E. B. (1985). Random fields associated with multiple points of the Brownian motion. *J. Funct. Anal.* **62** 397-434

[68] Dynkin, E. B. (1988). Regularized self-intersection local times of planar Brownian motion. *Ann. Probab.* **16** 58–74.

[69] Edwards , R. E. (1979) *Fourier Series An Modern Introduction* **1** 2nd Edition, Springer New York.

[70] Edwards , R. E. (1982) *Fourier Series An Modern Introduction.* **2** 2nd Edition, Springer New York.

[71] Ellis, R. S. (1984). Large deviations for a general class of random vector *Ann. Probab.* **12** 1-12.

[72] Ellis, R. S. (1985). *Entropy, Large Deviations and Statistical Mechanics* Springer-Verlag, New-York

[73] Feng, J. and Kurtz, T. G. *Large Deviations for Stochastic Processes* Mathematical Surveys and Monographs **131** Amer. Math. Soc.

[74] Fernández, R., Fröhlich, J. and Sokal, A. D. (1992).*Random Walks, Critical Phenomena, and Triviality in quantum field theory.* Springer, New York.

[75] Fleischmann, K., Mörters, P. and Wachtel, V. Moderate deviations for random walk in random scenery. *Stoch. Proc. Appl.* (to appear).

[76] Freidlin, M. I. and Wentzell, A. D. (1984). *Random Perturbations of Dynamical Systems.* Springer-Verlag New York

[77] Fukushima, M. (1980). *Dirichlet Forms and Markov Processes.* North-Holland/Kodanska, Amsterdam.

[78] Ganternt, R., König, W. and Shi, Z. (2007). Annealed deviations of random walk in random scenery. *Ann. Inst. H. Poincaré* **43** 47-76.

[79] Gantert, N., van der Hofstad, R. and König, W. (2006). Deviations of a random walk in a random scenery with stretched exponential tails. *Stochastic Process. Appl.* **116** 480–492.

[80] Gärtner, J. (1977). On large deviations from the invariant measures. *Theor. Probab. Appl.* **22** 24-39.

[81] Geman, D. and Horowitz, J. (1980). Occupation densities. *Ann. Probab.* **8** 1-67.

[82] Geman, D., Horowitz, J. and Rosen, J. (1984). A local time analysis of intersections of Brownian paths in the plane. *Ann. Probab.* **12** 86-107.

[83] Hamana, Y. (1997). The fluctuation result for the multiple point range of two dimensional recurrent random walks. *Ann. Probab.* **25** 598-639.

[84] Hamana, Y. (1998). An almost sure invariance principle for the range of random walks. *Stochastic Proc. Appl.* **78** 131-143.

[85] Hamana, Y. (2001). Asymptotics of the moment generating function for the range of random walks. *J. Theor. Probab.* **14** 189-197.

[86] Hamana, Y. and Kesten, H. (2001). A large-deviation result for the range of random walk and for the Wiener sausage. *Probab. Theory Related Fields* **120** 183-208.

[87] Hamana, Y. and Kesten, H. (2002). Large deviations for the range of an integer valued random walk. *Ann. Inst. H. Poincaré Probab. Statist.* **38** 17-58.

[88] van der Hofstad, R., den Hollander, F. and König, W. (2003). Large deviations for the one-dimensional Edwards model. *Ann. Probab.* **31** 2003–2039.

[89] van der Hofstad, R. and König, W. (2001). A survey of one-dimensional random polymers. *J. Statist. Phys.* **103** 915–944.

[90] van der Hoftad, R., König, W. and Mörters, P. (2006). The universality classes in the parabolic Anderson model. *Comm. Math. Phys.* **267** 307-353.

[91] den Hollander, F. (1996). Random polymers. *Stat. Nederl* **50** 136-145.

[92] Hu, Y. and Nualart, D. (2005). Renormalized self-intersection local time for fractonal Brownian motion. *Ann. Probab.* **33** 948-983.

[93] Ito, K. and McKean, H. P., jr (1988). *Diffusion Processes and Their Sample Paths* Springer New York

[94] Jain, N. C. and Orey, S. (1968). On the range of random walk. *Israel J. Math.* **6** 373-380.

[95] Jain, N.C. and Pruitt, W.E. (1971). The range of transient random walk. *J. Anal. Math.* **24** 369-393.

[96] Jain, N.C. and Pruitt, W.E. (1972). The law of the iterated logarithm for the range of random walk. *Ann. Math. Statist.* **43** 1692-1697.

[97] Jain, N.C. and Pruitt, W.E. (1983). Asymptotic behavior of the local time of a recurrent random walk. *Ann. Probab.* **11**64-85.

[98] Kac, M. (1951). On some connections between probability theory and differential and integral equations. *Proc. Second Berkeley Symposium, L. M. Le Cam and J. Neyman editors.* University of California Press, Berkeley, pages 189-215.

[99] Kantor, Y. and Kardar, M. (1991). Polymers with self-interactions. *Europhys. Lett.* **14** 421-426.

[100] Kesten, H. and Spitzer, F. (1979). A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete* **50** 2-25.

[101] Khanin, K. M., Mazel, A. E., Sholosman, S. B. and Sinai, Ya. G. (1994). Loop condensation effect in the behavior of random walks. *Markov process and their applications (Freidlin, edu), Progr. Probab.* **34**, Birkhäuser, Boston, p.167-184.

[102] Khoshnevisan, D. (2002). *Multiparameter Processes — An Introduction to Random fields.* Springer, New York.

[103] Khoshnevisan, D. (2003). Intersection of Brownian motions. *Exposition. Math.* **21** 97-114.

[104] Khoshnevisan, D. and Lewis, T. M. (1998). A law of the iterated logarithm for stable processes in random scenery. *Stochastic Proc. Appl.* **74** 89-121.

[105] Khoshnevisan, D., Xiao, Y. and Zhong, Y. (2003a). Local times of additive Lévy processes. *Stoch. Proc. Their Appl.* **104** 193-216.

[106] Khoshnevisan, D., Xiao, Y. and Zhong, Y. (2003b). Measuring the range of an additive Lévy process. *Ann. Probab.* **31** 1097–1141..

[107] König, W. and Mörters, P. (2002). Brownian intersection local times: Upper tail asymptotics and thick points. *Ann Probab.* **30** 1605-1656.

[108] König, W. and Mörters, P. (2006). Brownian intersection local times: exponential moments and law of large masses. *Trans. Amer. Math. Soc.* **358** 1223–1255.

[109] Lawler, G. F. (1991). *Intersections of Random Walks, Probability and Its applications.* Birkhäusen Boston

[110] Lawler, G. F. (2005). *Conformally Invariant Processes in the Plane. Mathematical Surveys and Monographs,* **114**. American Mathematical Society, Providence, RI.

[111] Lawler, G. F., Schramm, O. and Werner, W. (2001). Values of Brownian intersection exponents. I. Half-plane exponents. *Acta Math.* **187** 237–273.

[112] Lawler, G. F., Schramm, O. and Werner, W. (2001). Values of Brownian intersection exponents. II. Plane exponents. *Acta Math.* **187** 275–308.

[113] Lawler, G. F., Schramm, O. and Werner, W. (2002). Values of Brownian intersection exponents. III. Two-sided exponents. *Ann. Inst. H. Poincaré Probab. Statist.* **38** 109–123.

[114] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach spaces.* Springer, Berlin.

[115] Le Gall, J-F. (1985). Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan. *Séminaire de Probabilités, XIX. Lecture Notes in Math* **1123**, 314-331. Springer, Berlin.

[116] Le Gall, J-F. (1986). Sur la saucisse de Wiener et les points multiples du mouvement brownien. *Ann. Probab.* **14** 1219-1244.

[117] Le Gall, J-F. (1986). Propriétés d'intersection des marches aléatoires. I. Convergence vers le temps local d'intersection. *Comm. Math. Phys.* **104**, 471-507,

[118] Le Gall, J-F. (1986). Propriétés d'intersection des marches aléatoires. II. Étude des cas critiques. *Comm. Math. Phys.* **104**, 509-528.

[119] Le Gall, J-F. (1987). Temps locaux d'intersection et points multiples des processus de Lévy. *Séminaire de Probabilités, XXI. Lecture Notes in Math* **1247**, 341-375. Springer, New York.

[120] Le Gall, J-F. (1988). Flutuation results for the Wiener sausuage. *Ann. Probab.* **16** 991-1018.

[121] Le Gall, J-F. (1990). Wiener sausuage and self-intersection local time. *J. Funct. Anal.* **88** 299-341.

[122] Le Gall, J-F. (1992). Some properties of planar Brownian motion. *École d'Été de Probabilités de Saint-Flour XX—1990. Lecture Notes in Math* **1527** 111-235. Springer, Berlin.

[123] Le Gall, J-F. (1994). Exponential moments for the renormalized self-intersection local time of planar Brownian motion. *Séminaire de Probabilités, XXVIII*, 172-180. Springer, Berlin.

[124] Le Gall, J-F. and Rosen, J. (1991). The range of stable random walks. *Ann. Probab.* **19** 650–705.

[125] Levine, H. A. (1980). An estimate for the best constant in a Sobolev inequality involving three integral norms. *Ann. Mat. Pura Appl.* **124** 181-125.

[126] Madras, N. and Slade, G. (1993). *The Self-avoiding Walk.* Birkhäuser, Boston.

[127] Mansmann, U. (1991). The free energy of the Dirac polaron, an explicit solution. *Stochastics & Stochastics Report.* **34** 93-125.

[128] Marcus, M. B.and Rosen, Jay, J. (1997). Laws of the iterated logarithm for intersections of random walks on Z^4. *Ann. Inst. H. Poincaré Probab. Statist.* **33** 37–63.

[129] Martanez, S. and Petritis, D. (1996). Thermodynamics of a Brownian bridge polymer model in a random environment. *J. Phys. A* **29** (1996), 1267–1279.

[130] Peres, Y. (2002). Brownian intersections, cover times and thick points via trees. *Proceedings of the International Congress of Mathematicians, Vol. III* Beijing, 73–78,

[131] Pinsky, R. G. (1995). *Positive Harmonic Functions and Diffusion.* Combridge University Press, Combridge.

[132] Remillard, B. (2000). Large deviations estimates for occupation time integrals of Brownian motion. *Stochastic models (Ottwa, ON)* 375-398, *CMS Cof. Proc.* **26** Amer. Soc. Providence. RI.

[133] Revuz, D. and Yor, M. (1994). *Continuous Martingale and Brownian Motion* 2nd Edition. Springer, Berlin.

[134] Révész, P. and Shi, Z. (2000). Strong approximation of spatial random walk in random scenery. *Stochastic Proc. Appl.* **88** 329-345.

[135] Robert, A. (1976). A short proof of the Fourier inversion formula. *Proc. Amer. Math. Soc.* **59** 287-288.

[136] Rogers, L. C. G. and Williams, D. (1987). *Diffusions, Markov processes, and martingales* **Vol. 2** *Ito Calculus* John Wiley & Sons, Inc., New York.

[137] Rogers, L. C. G. and Williams, D. (1994). *Diffusions, Markov processes, and martingales* **Vol.1**. *Foundations* John Wiley & Sons, Inc., New York.

[138] Rosen, J. (1983). A local time approach to the self-intersections of Brownian paths in space. *Comm. Math. Physics* **88** 327-338.

[139] Rosen, J. (1984). Self-intersections of random fields. *Ann. Probab.* **12** 108-119.

[140] Rosen, J. (1986). Tanaka formula and renormalization for intersections of planar Brownian motion. *Ann. Probab.* **14** 1245-1251.

[141] Rosen, J. (1986). A renormalized local time for multiple intersections of planar Brownian motion. *Séminaire de Probabilités, XX, 1984/85, Lecture Notes in Math.,* **1204** 515-531.

[142] Rosen, J. (1987). Joint continuity of the intersection local time of Markov processes. *Ann. Probab.* **12** 108-119.

[143] Rosen, J. (1990). Random walks and intersection local time. *Ann. Probab.* **18** 959-977.

[144] Rosen, J. (1997). Laws of the iterated logarithm for triple intersections of three-dimensional random walks. *Electron. J. Probab.* **2** 2, 1–32

[145] Rosen, J. and Yor, M. (1991). Tanaka formulae and renormalization for triple intersections of Brownian motion in the plane. *Ann. Probab.* **19** 142-159.

[146] Saloff-Coste, L. (2002). *Aspects of Sobolev-Type Inequalities* London Math. Soc. LNS Vol **289**, Cambridge University Press, Cambridge.

[147] Spitzer, F. (1964). *Principles of Random Walk*, Van Nostrand, Princeton, New Jersey.

[148] Stroock, D. W. (1984). *An Introduction to the Theory of Large deviations.* Springer Berlin.

[149] Sznitman, A. S. (1990). Long time asymptotics for the shrinking Wiener sausage. *Comm. Pure & Appl. Math.* **43** 809-820.

[150] Sznitman, A. S. (1998). *Brownian Motion, Obstacles and Random media.* Springer Berlin

[151] Uchiyama, K. (1998). Wiener's test for random walks with mean zero and finite variance. *Ann. Probab.* **26** 368-376.

[152] Varadhan, S. R. S. (1984). *Large Deviations and Applications.* SIAM, Philadelphia.

[153] Varadhan, S. R. S. (1969). Appendix to "Euclidean quantum field theory", by K. Symmanzik. *Local Quantum theory* (R. Jost ed). Academic New York.

[154] Varadhan, S. R. S. (2008). Large deviations. *Ann. Probab.* **36** 397-419.

[155] Vanderzande, C. (1998) *Lattice models of Polymers.* Cambridge University Press, Cambridge.

[156] Watson, G. N. (1995) *A Treatise on the Theory of Bessel Functions*, Second Edition, (Cambridge University Press. Combridge.

[157] Weinstein, M. I. (1983). Nonlinear Shrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.* **87** 567-576.

[158] Westwater, J. (1980). On Edwards' model for long polymer chains. *Comm. Math. Phys.* **72** 131-174

[159] Westwater, J. (1981). On Edwards' model for polymer chains. II. The self-consistent potential. *Comm. Math. Phys.* **79** 53-73

[160] Westwater, J. (1982). On Edwards' model for polymer chains. III. Borel summability. *Comm. Math. Phys.* **84** 459-470.

[161] Wolpert, R. (1978). Wiener path intersections and local time. *J. Funct. Anal.* **30** 329-340.

[162] Xiao, Y. (2006). Properties of local nondeterminism of Gaussian and stable random fields and their applications. (preprint).

[163] Yor, M. (1985). Compléments aux formules de Tanaka-Rosen. *Séminaire de Probabilités XIX. Lecture Notes in Math.* **1123** 332–349. Springer, Berlin.

[164] Yor, M. (1985). Renormalisation et convergence en loi pour les temps locaux d'intersection du mouvement brownien dans $\mathbb{R}^3$. *Séminaire de Probabilités XIX. Lecture Notes in Math.* **1123** 350–365. Springer, Berlin.

[165] Yor, M. (1986). Précisions sur l'existence et la continuité des temps locaux d'intersection du mouvement brownien dans $\mathbb{R}^d$ *Séminaire de Probabilités XX. Lecture Notes in Math.* **1204** 532-541. Springer, Berlin.

[166] Yor, M. (1986). Sur la représentation comme intégrales stochastiques des temps d'occupation du mouvement brownien dans $\mathbb{R}^d$. *Séminaire de Probabilités XX. Lecture Notes in Math.* **1204** 543-552. Springer, Berlin.

[167] Yor, M. Remarques sur l'intégrabilité du temps local d'intersection renormalisé. (unpublished).

[168] Yosida, K. (1966). *Functional Analysis.* Springer Berlin

[169] Ziemer, W. P. (1989). *Weakly Differentiable Functions.* Springer, New York

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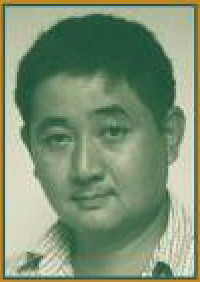
The material covered in this book involves important and nontrivial results in contemporary probability theory motivated by polymer models, as well as other topics of importance in physics and chemistry. The development carefully provides the basic definitions of mutual intersection and self-intersection local times for Brownian motions and the accompanying large deviation results. The book then proceeds to the analogues of these concepts and results for random walks on lattices of R^d. This includes suitable integrability and large deviation results for these models and some applications. Moreover, the notes and comments at the end of the chapters provide interesting remarks and references to various related results, as well as a good number of exercises. The author provides a beautiful development of these subtle topics at a level accessible to advanced graduate students.



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